IBM Research

Sieving for closest lattice vectors (with preprocessing)

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Lattices

What is a lattice?

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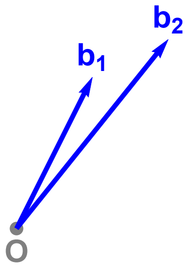

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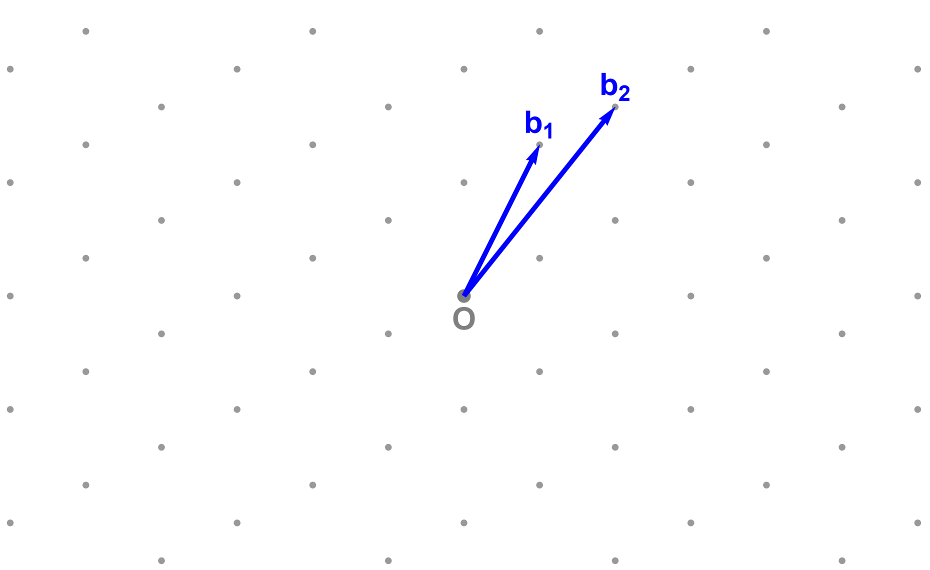
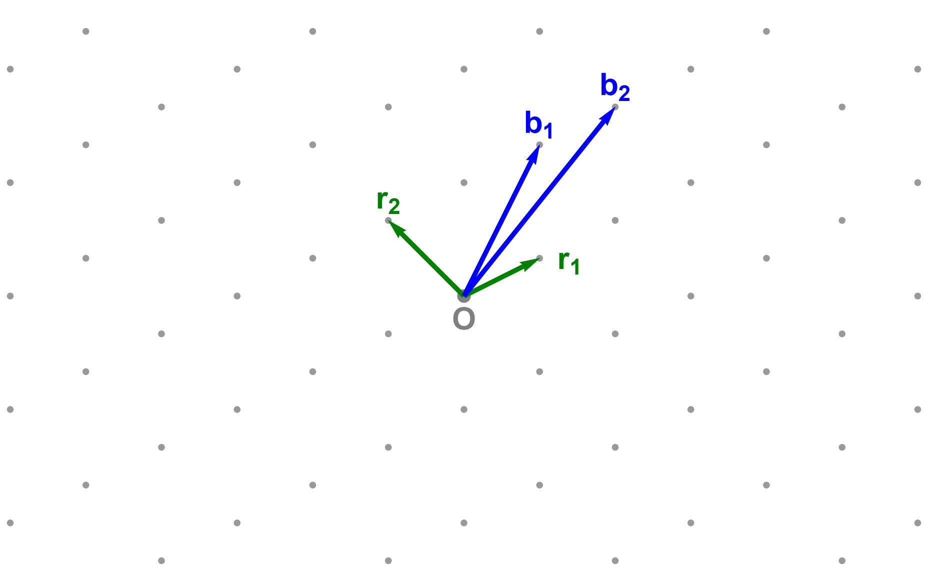


Lattices

Lattice basis reduction



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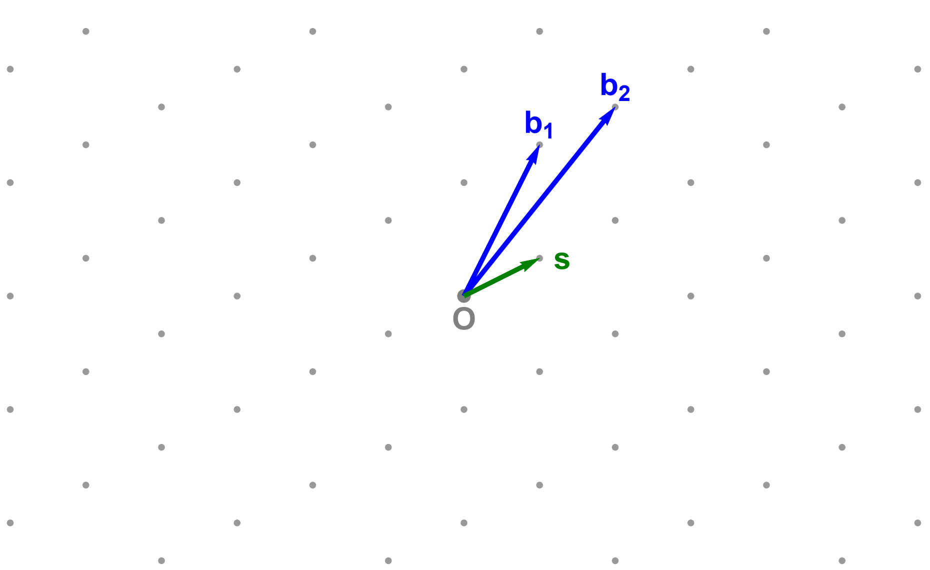
Shortest Vector Problem (SVP)



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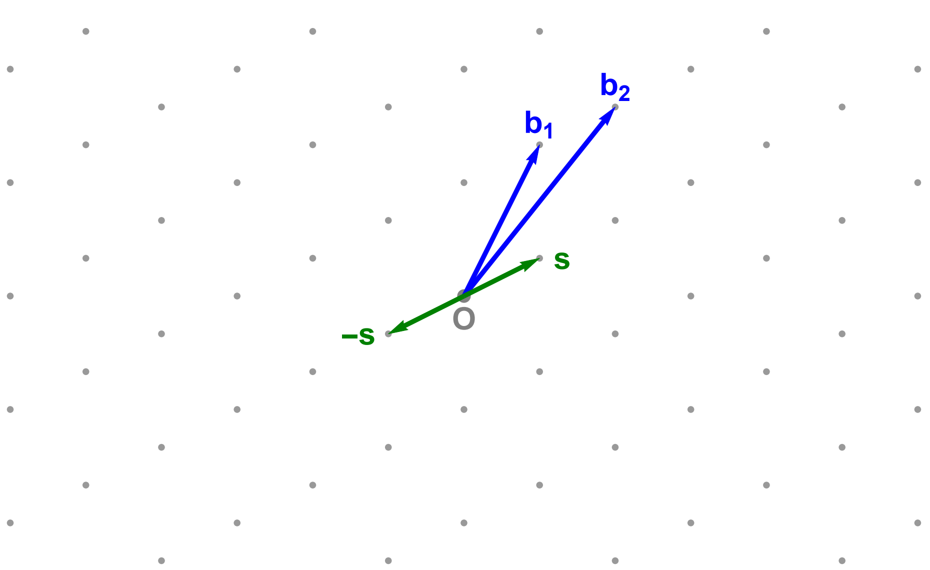

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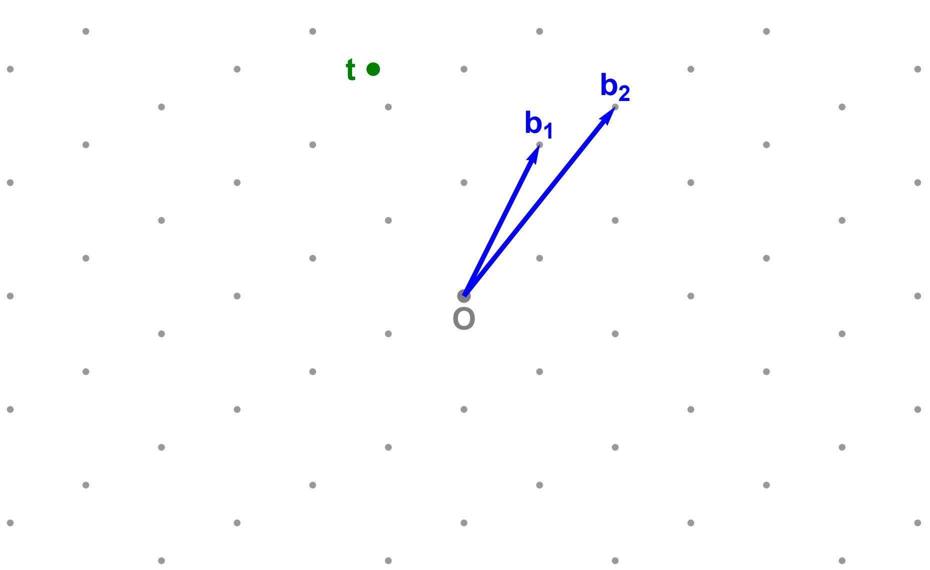

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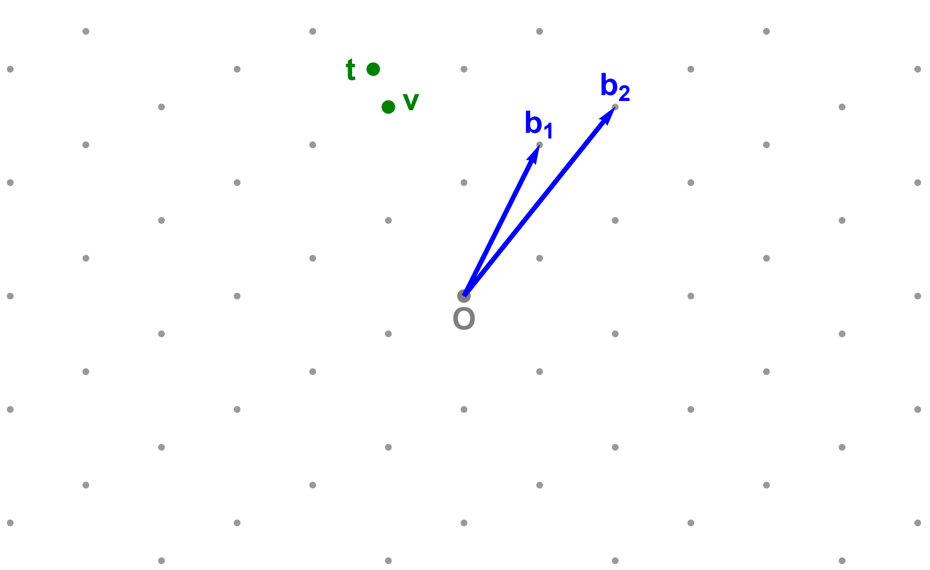

Outline

Sieving for SVP

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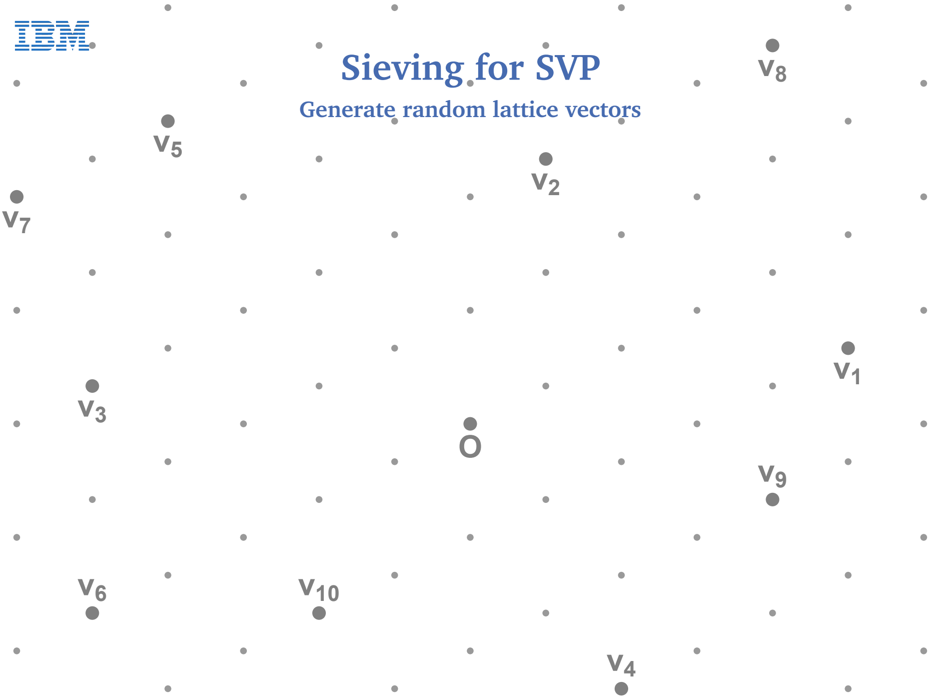
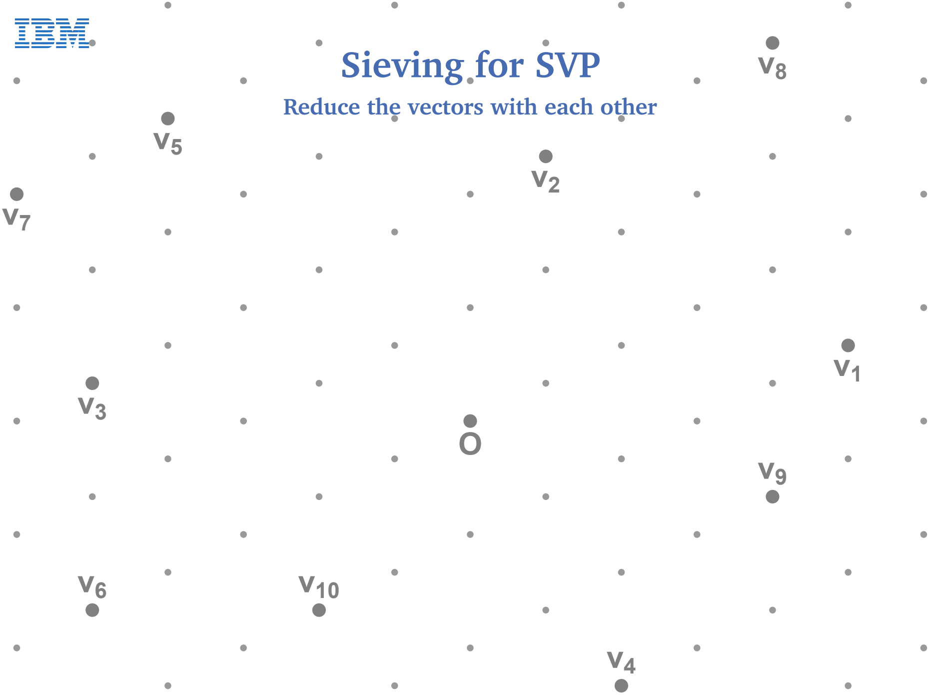


Sieving for SVP

Reduce the vectors with each other



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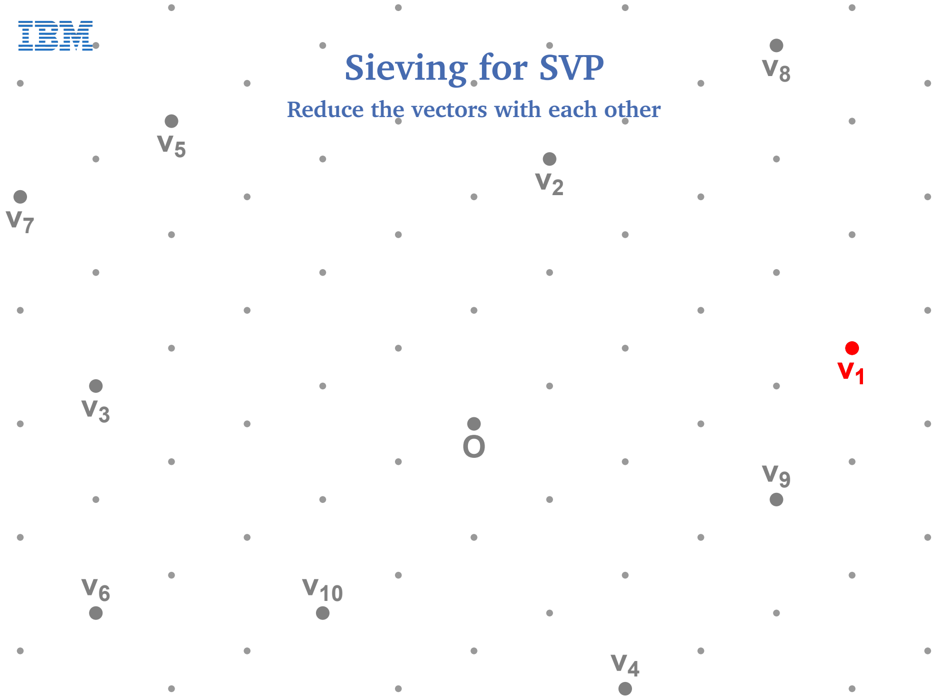

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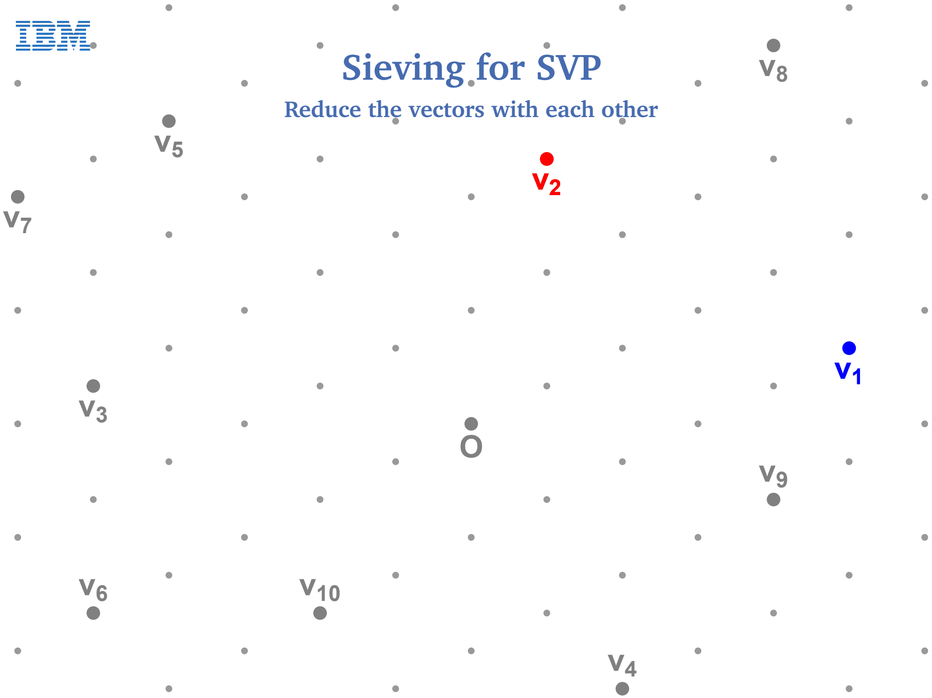

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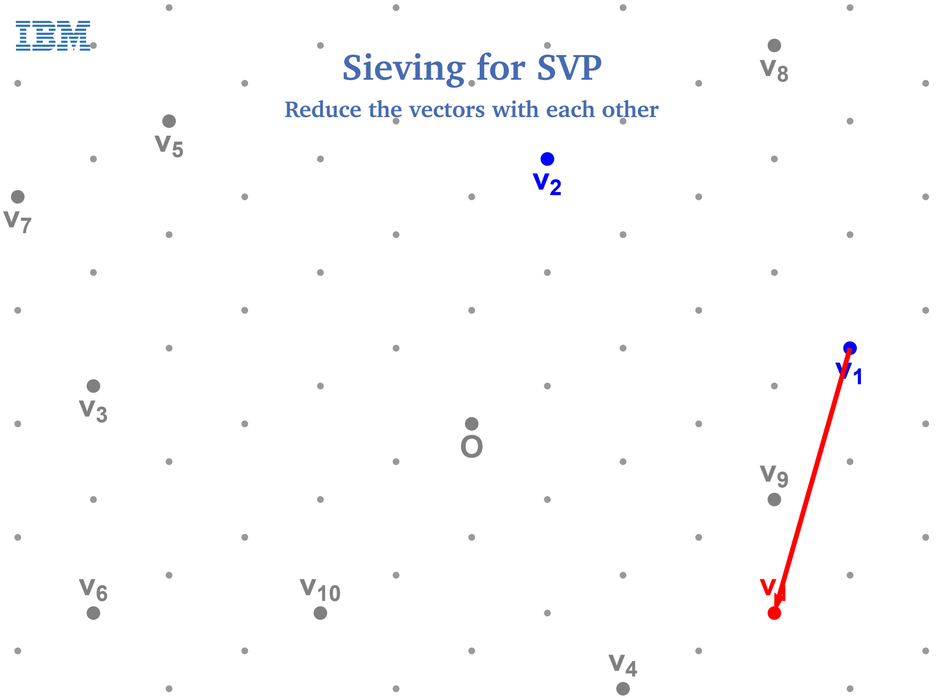

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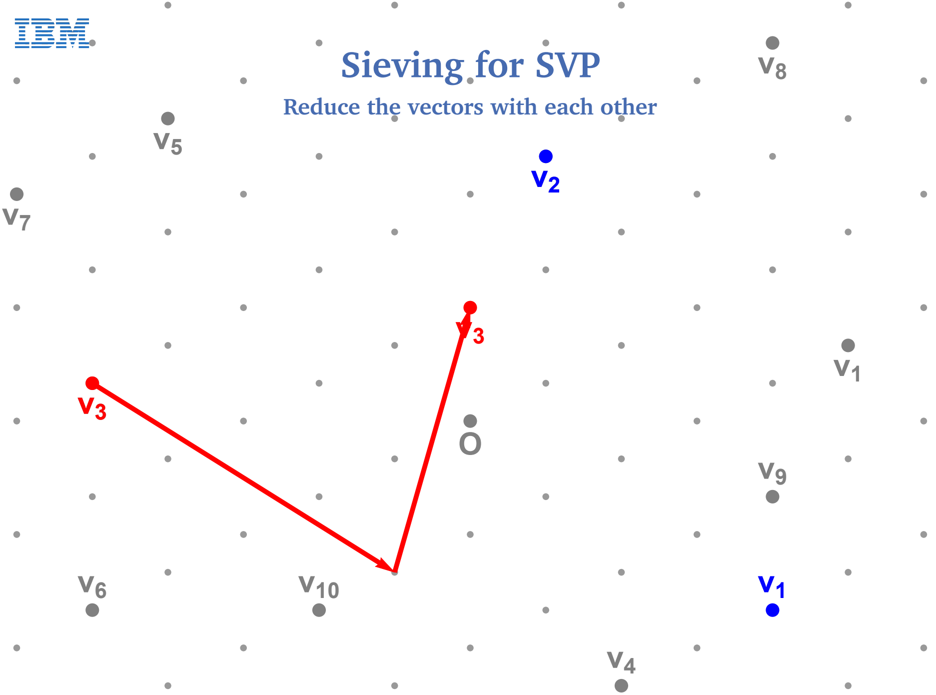

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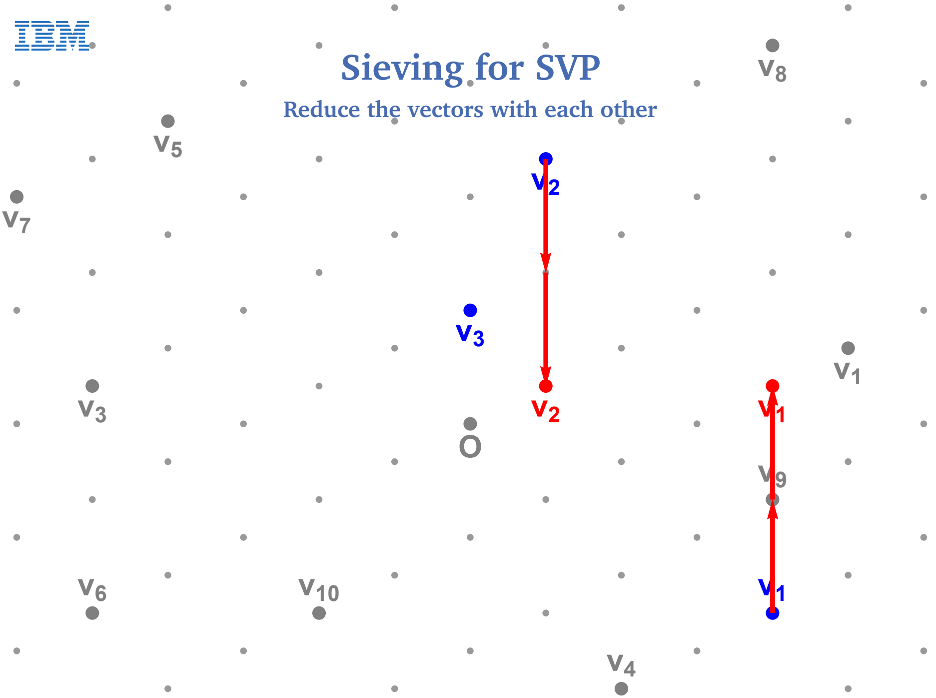

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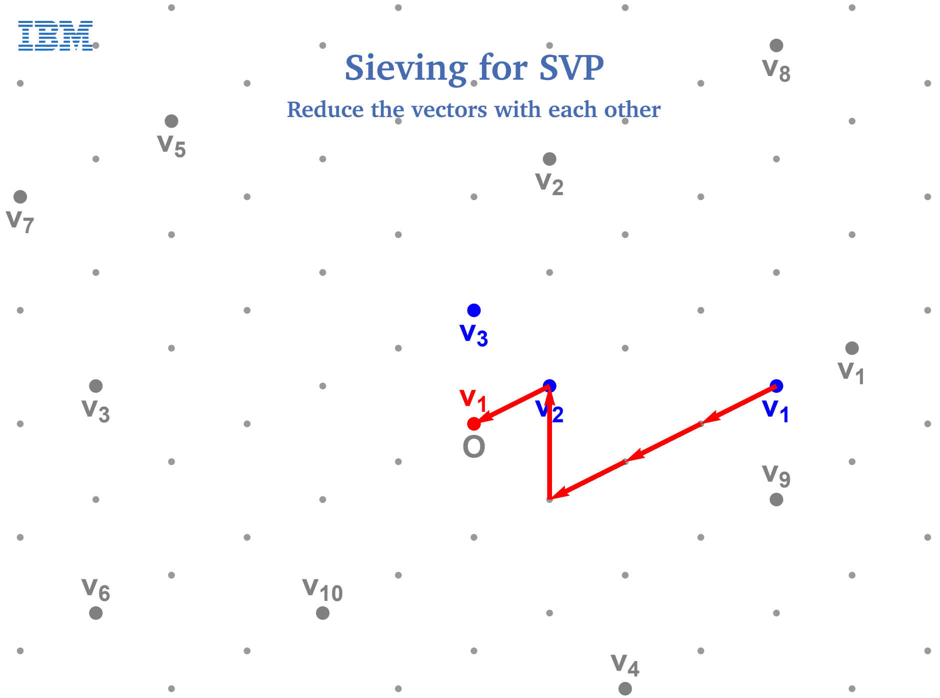

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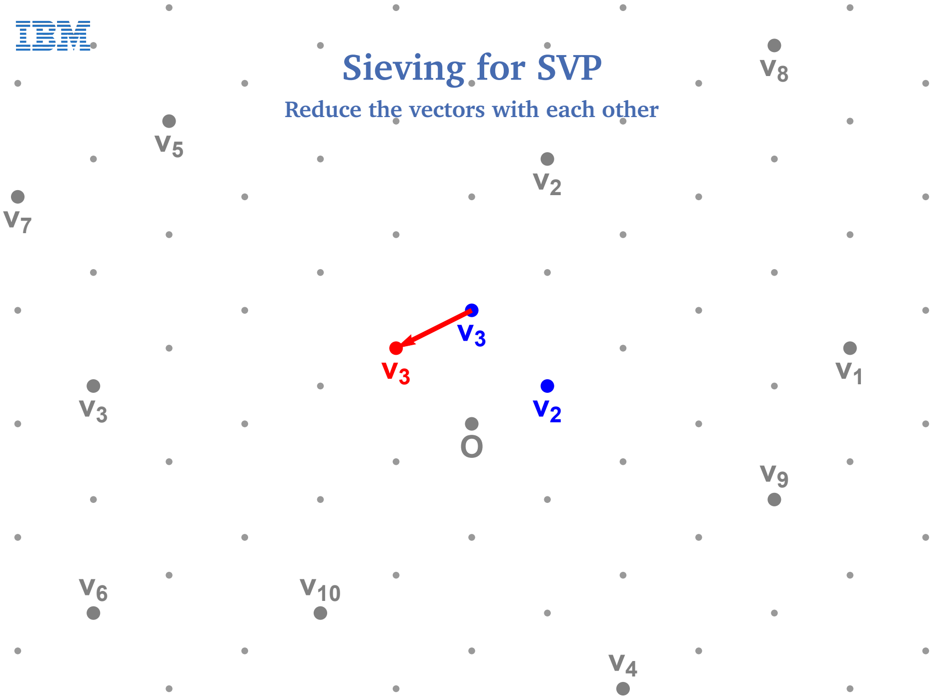

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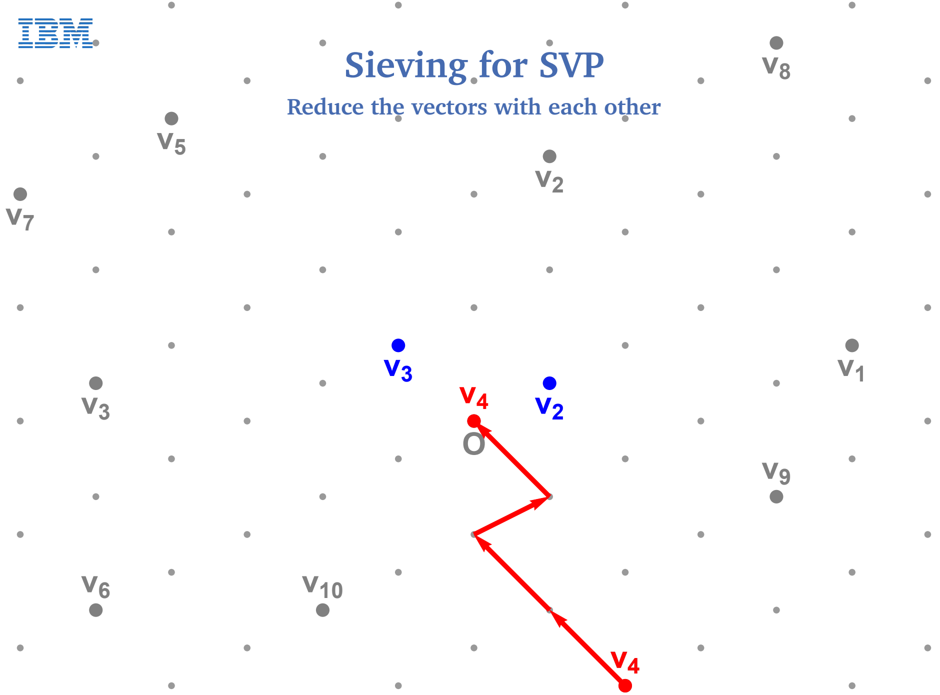

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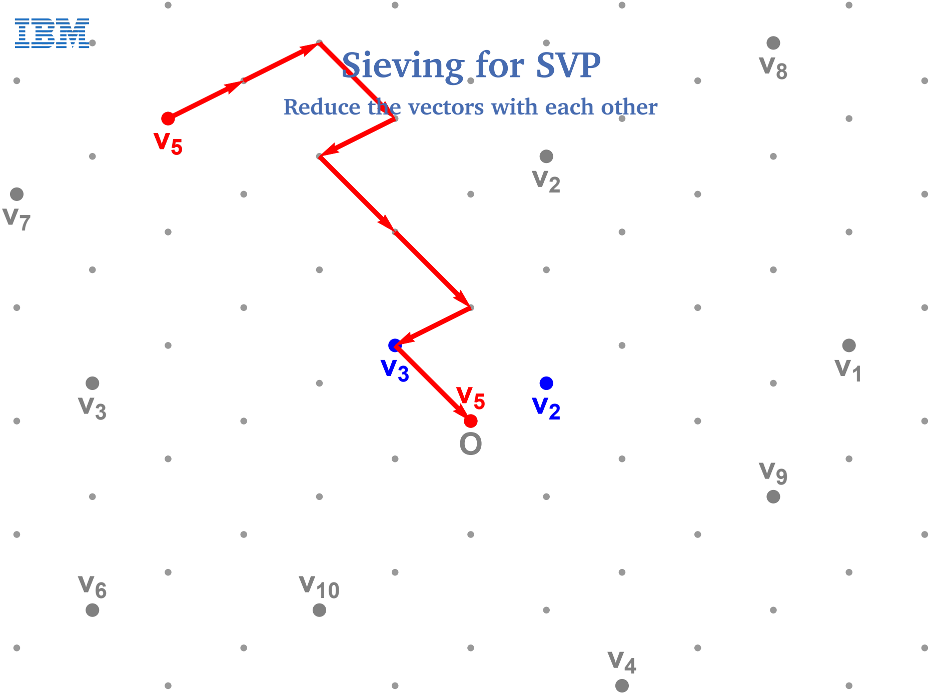

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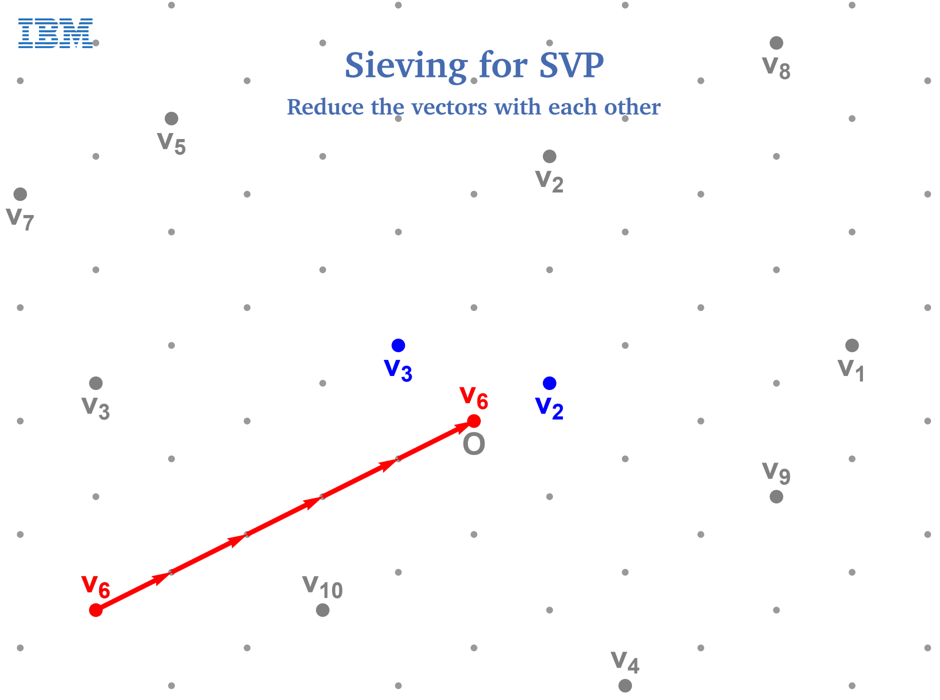

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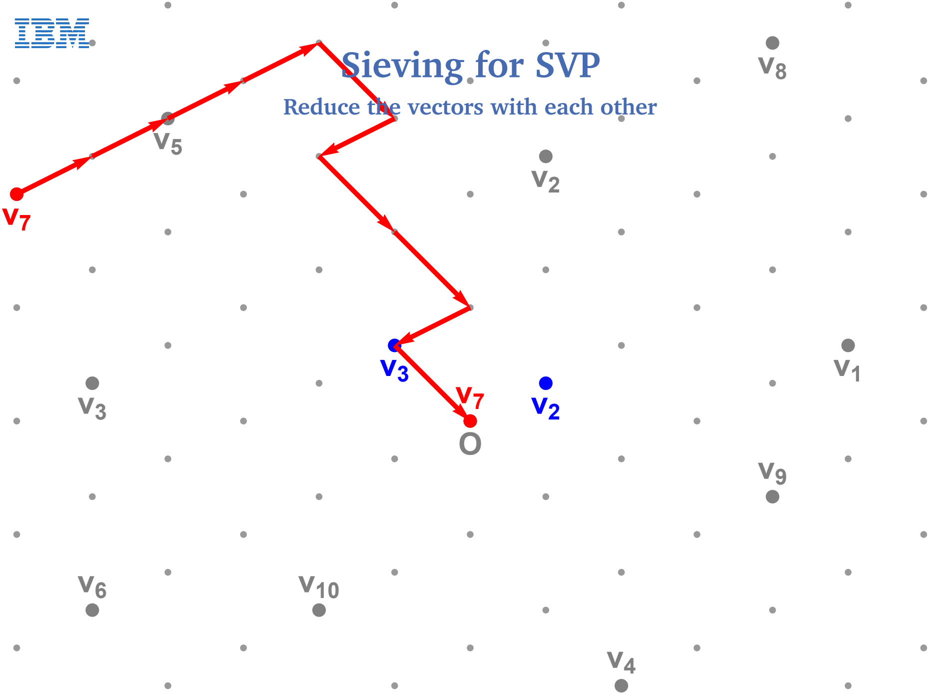

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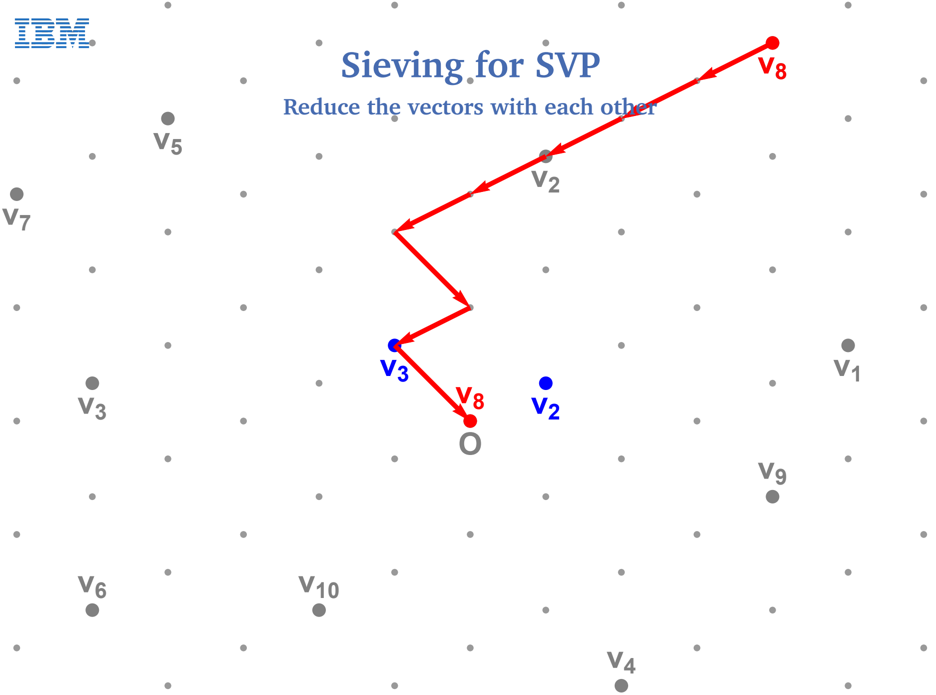

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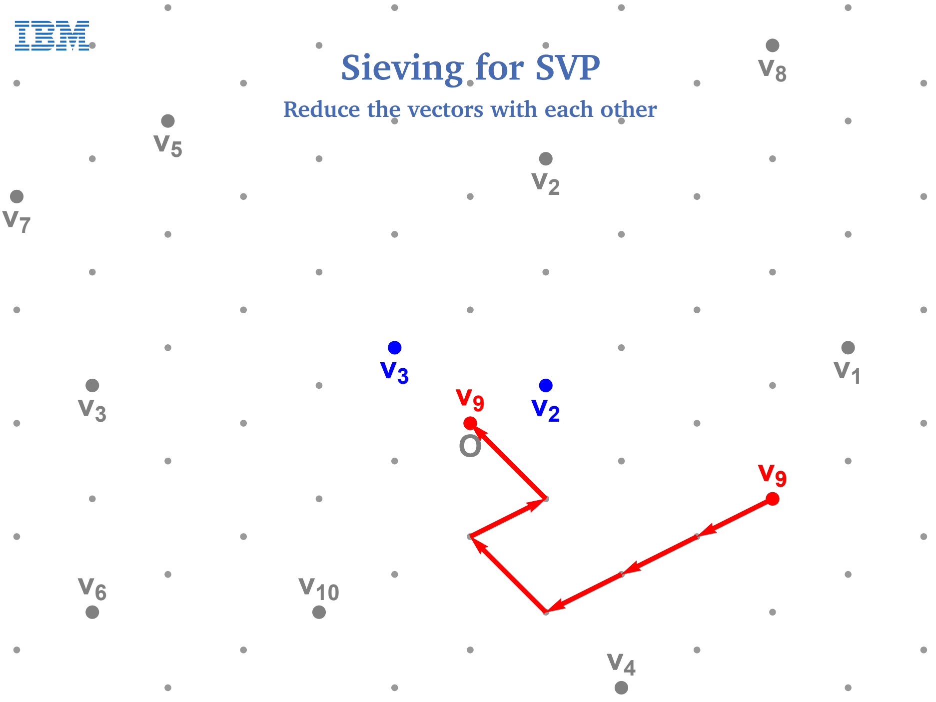

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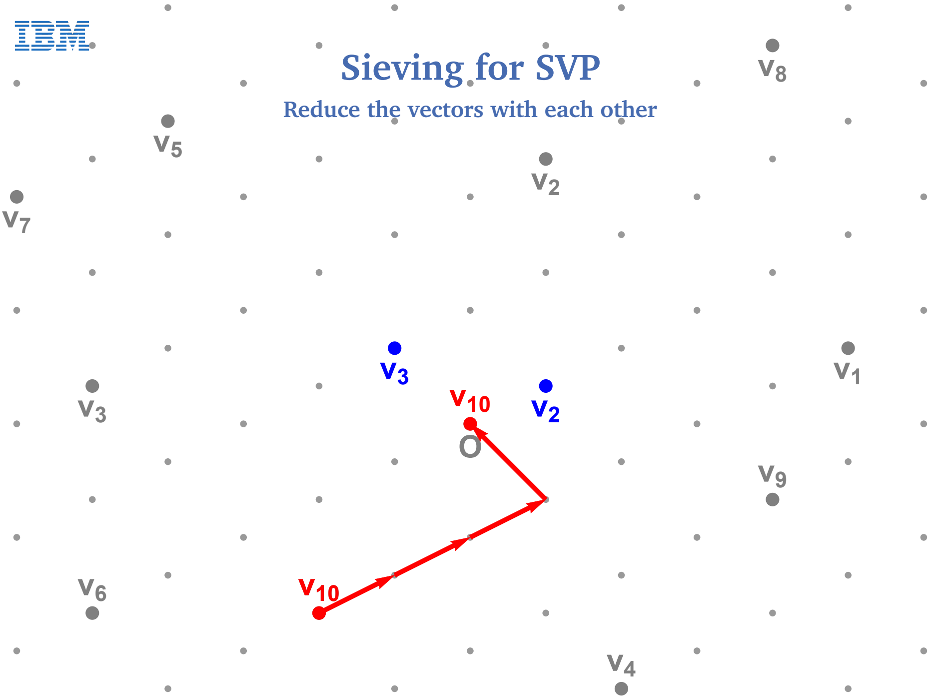

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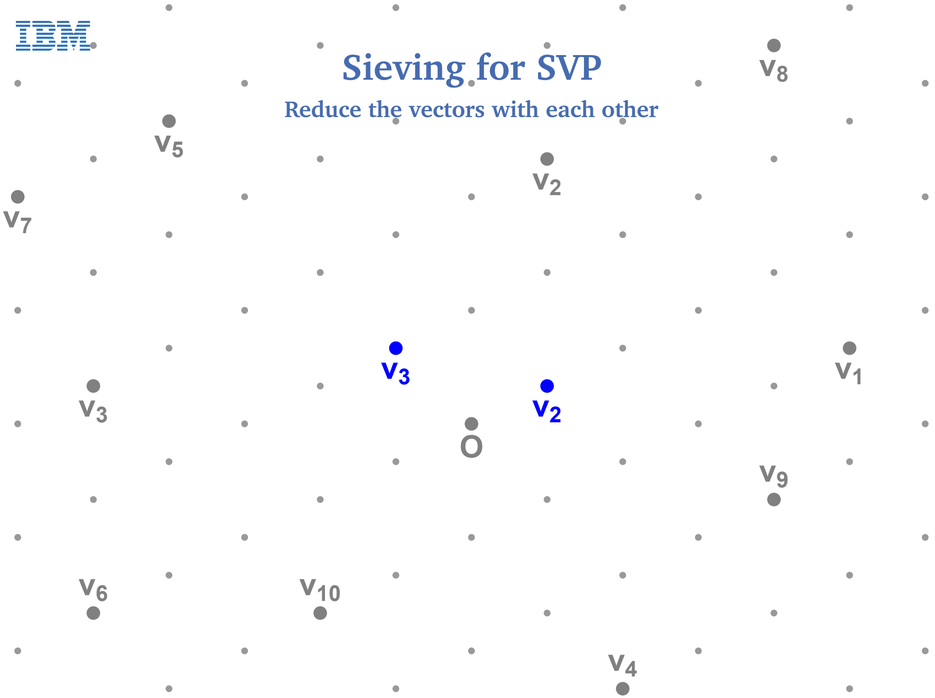

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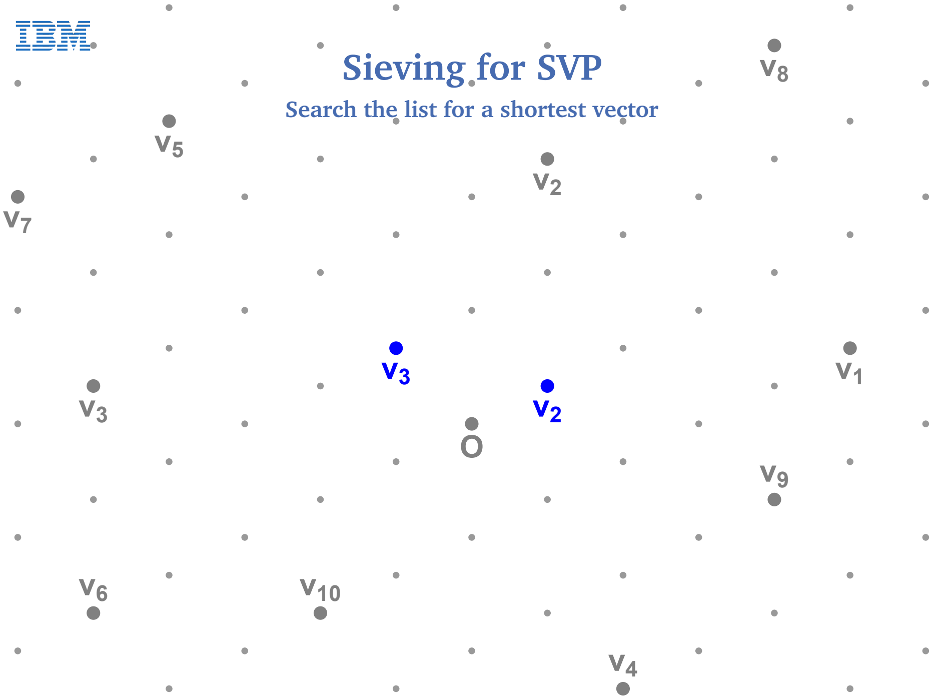

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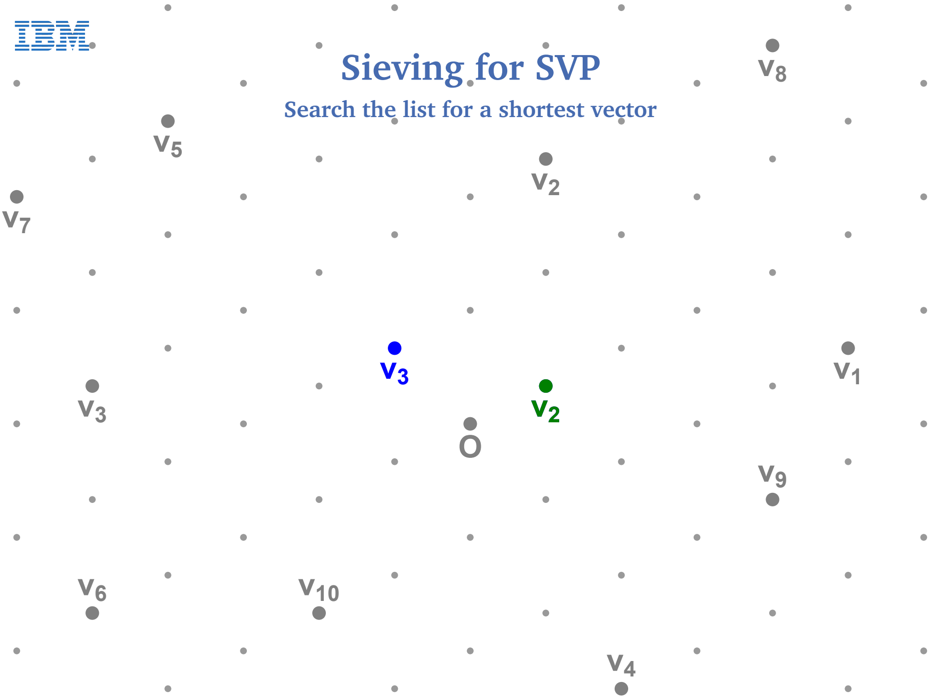
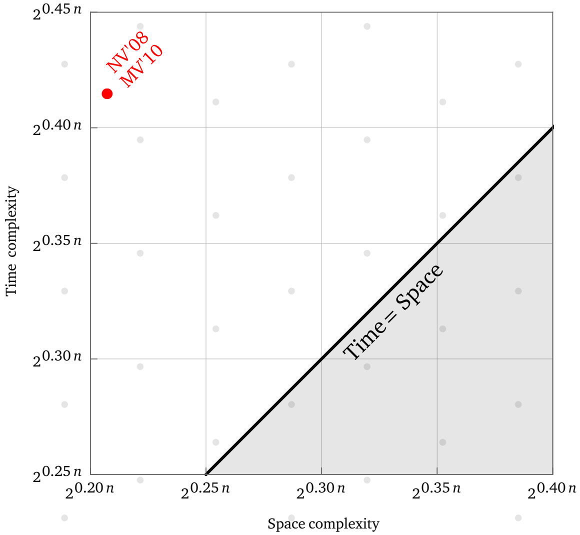


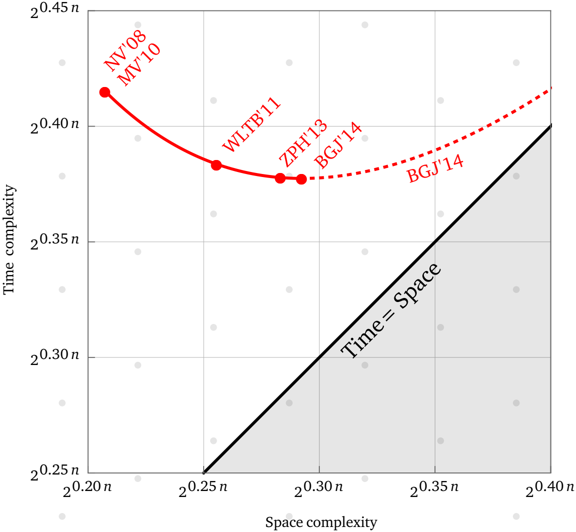
Sieving for SVP

The GaussSieve and Nguyen-Vidick sieve



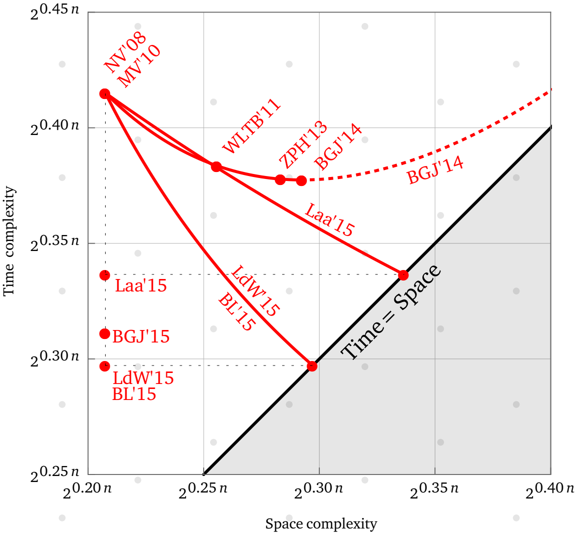
Sieving for SVP

Leveled sieving approaches



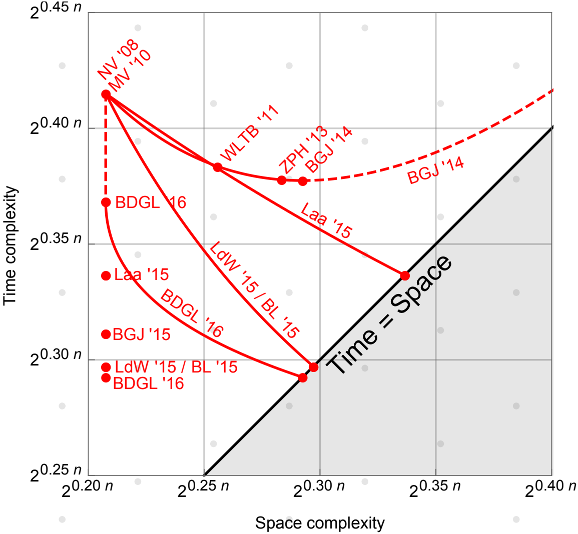
Sieving for SVP

Locality-Sensitive Hashing (LSH)



Sieving for SVP

Locality-Sensitive Filters (LSF)



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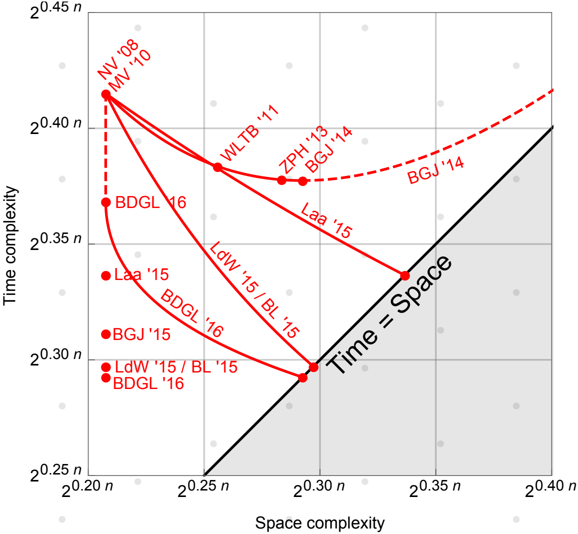
Space/time trade-offs



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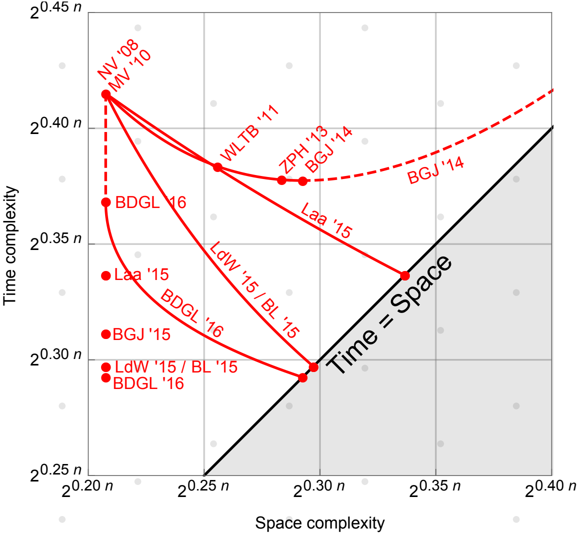

Sieving for CVP

- Intuitively, $\mathrm{CVP}_n \approx \mathrm{SVP}_{n+1}$ [Kan87]

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- Can also directly modify sieving to solve CVP
- Costs of CVP_n factor 2 more than SVP_n

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Run a GaussSieve as preprocessing

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Reduce the target vectors with the list

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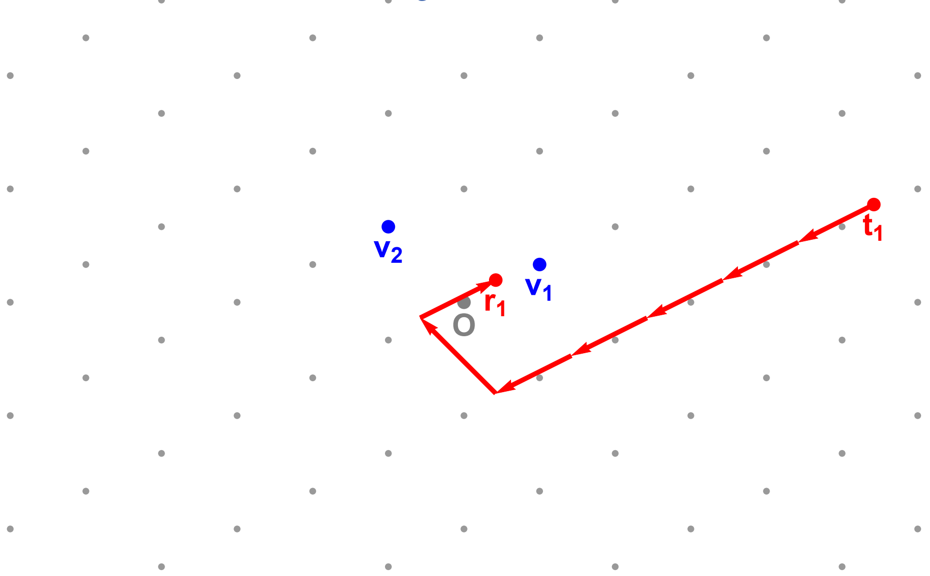

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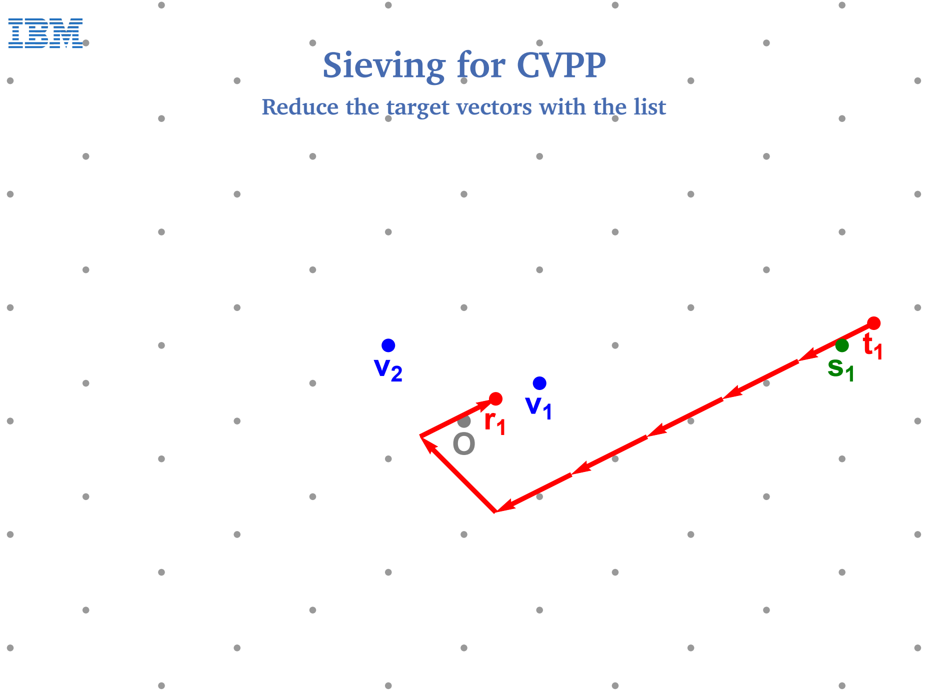

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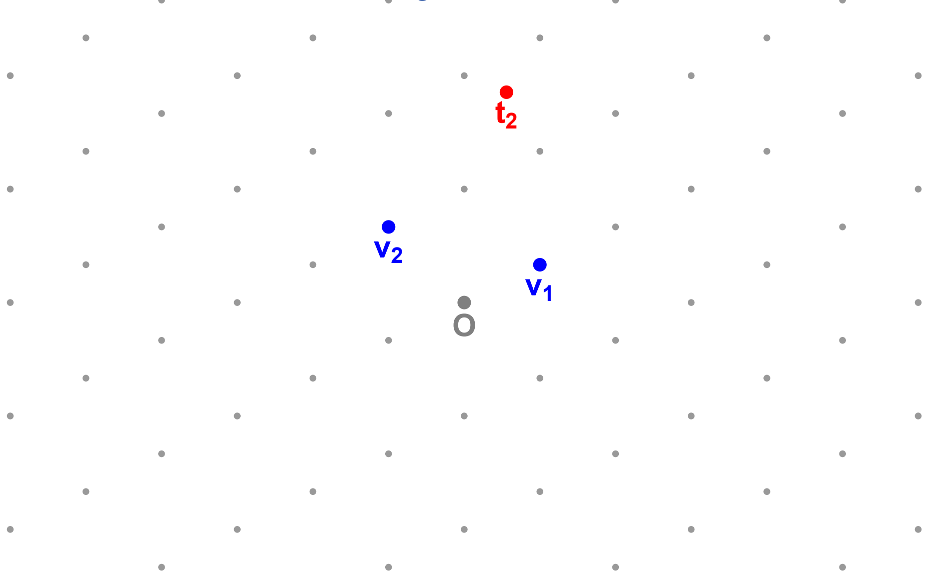

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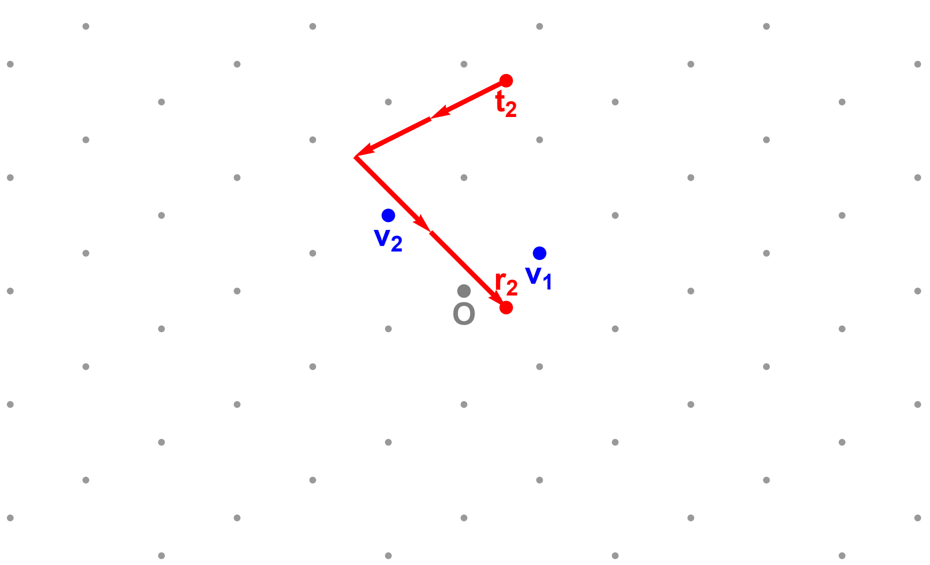

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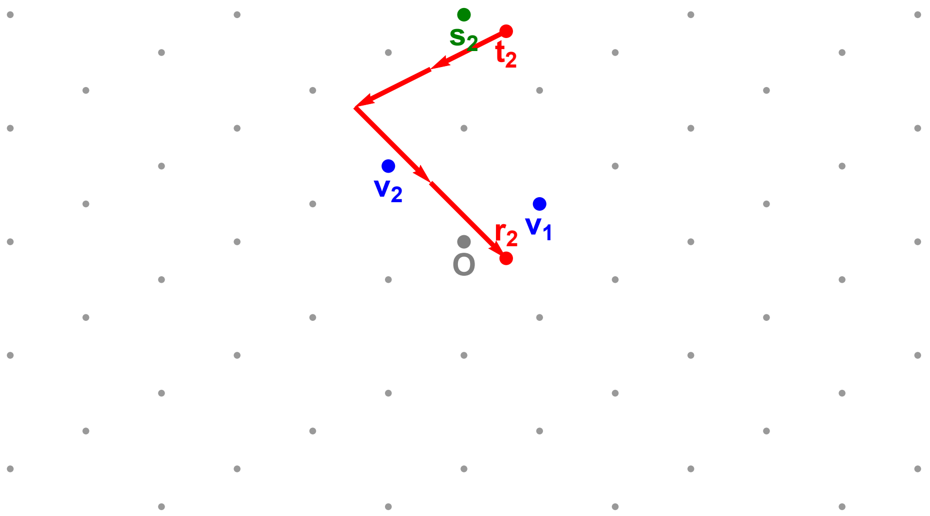

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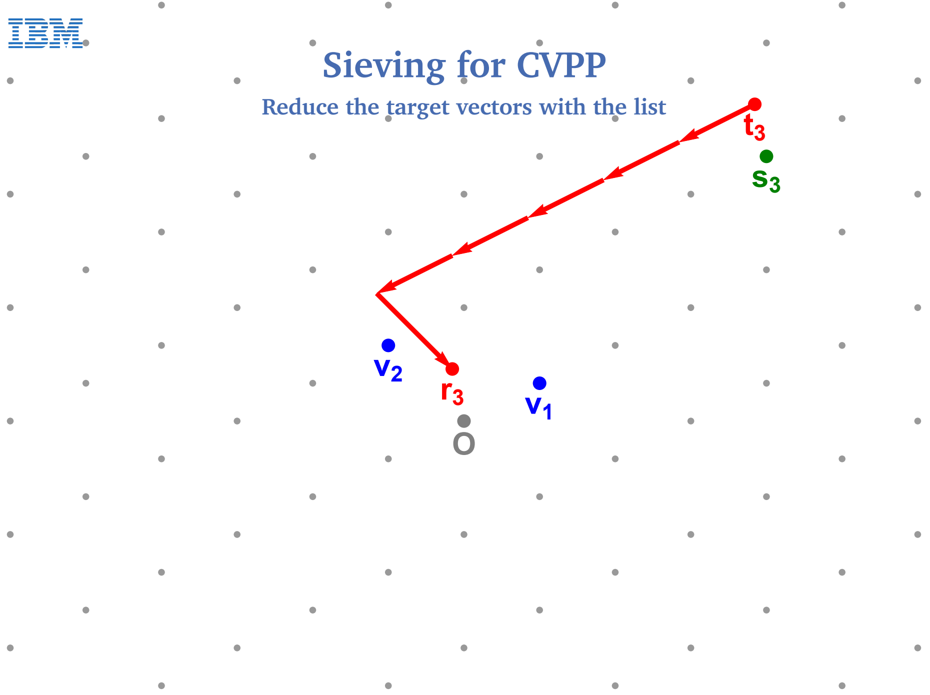

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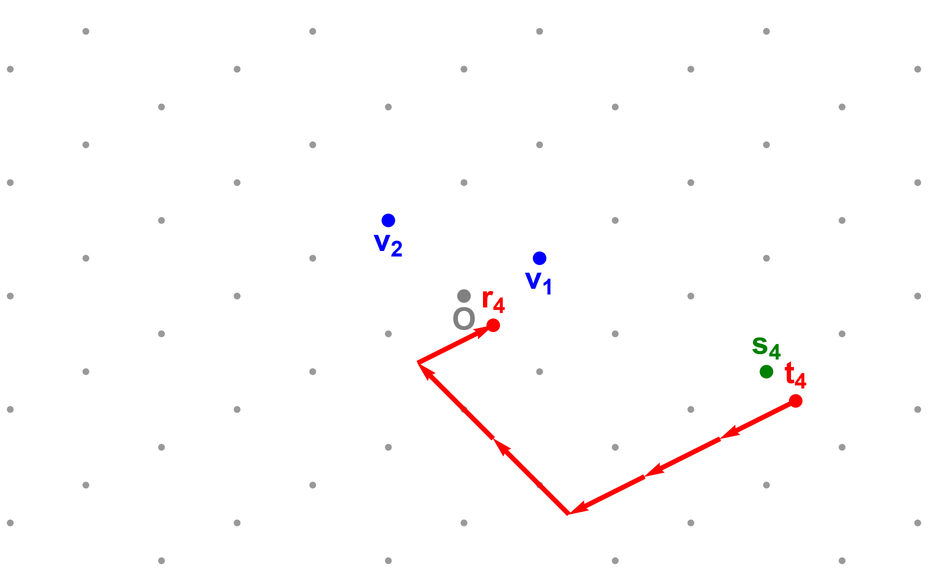

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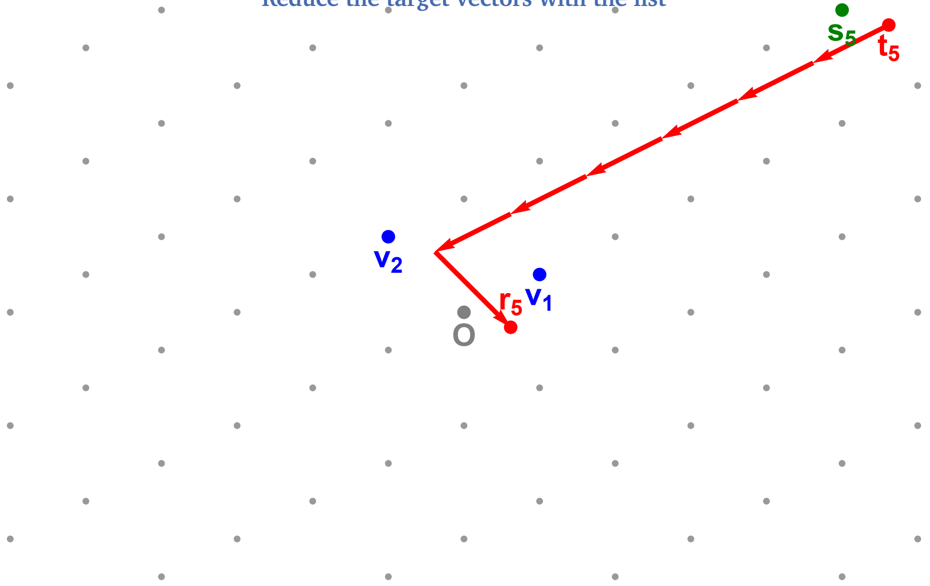

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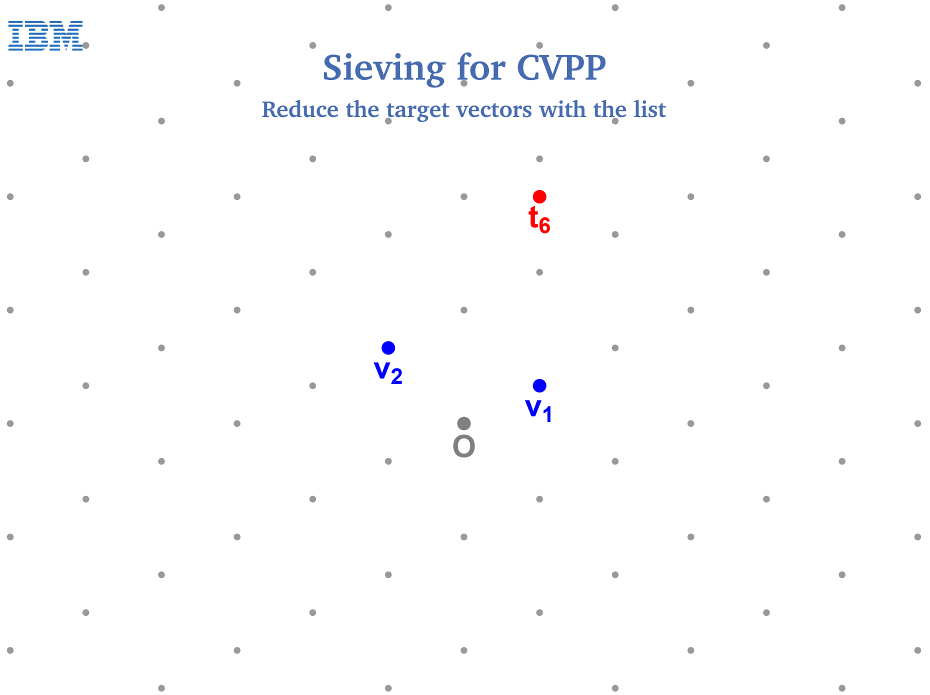

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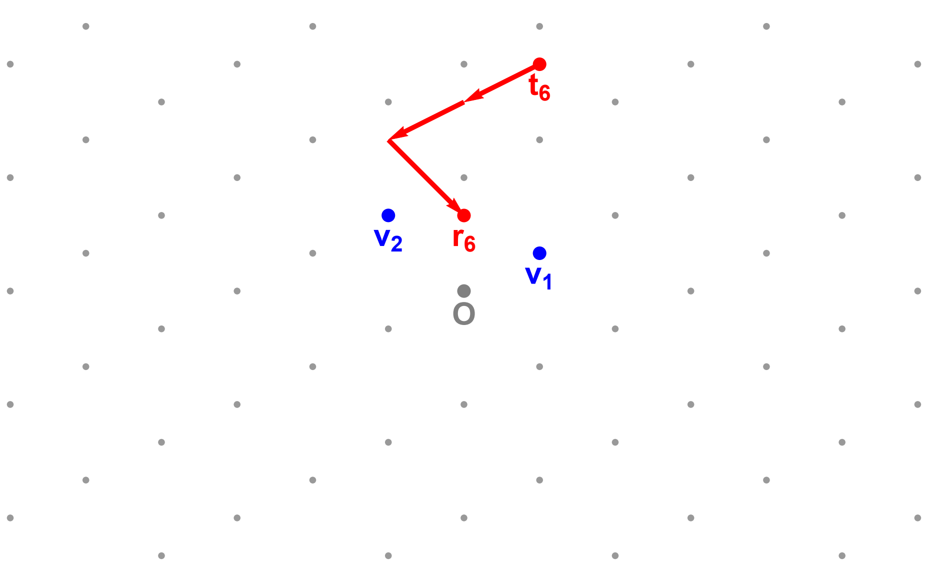

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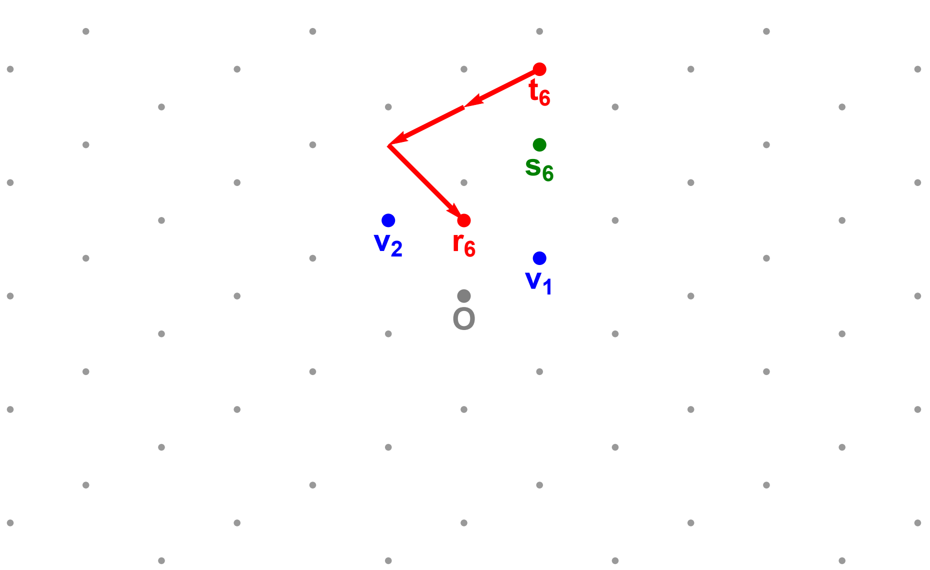

Sieving for CVPP

Relation with the Voronoi cell

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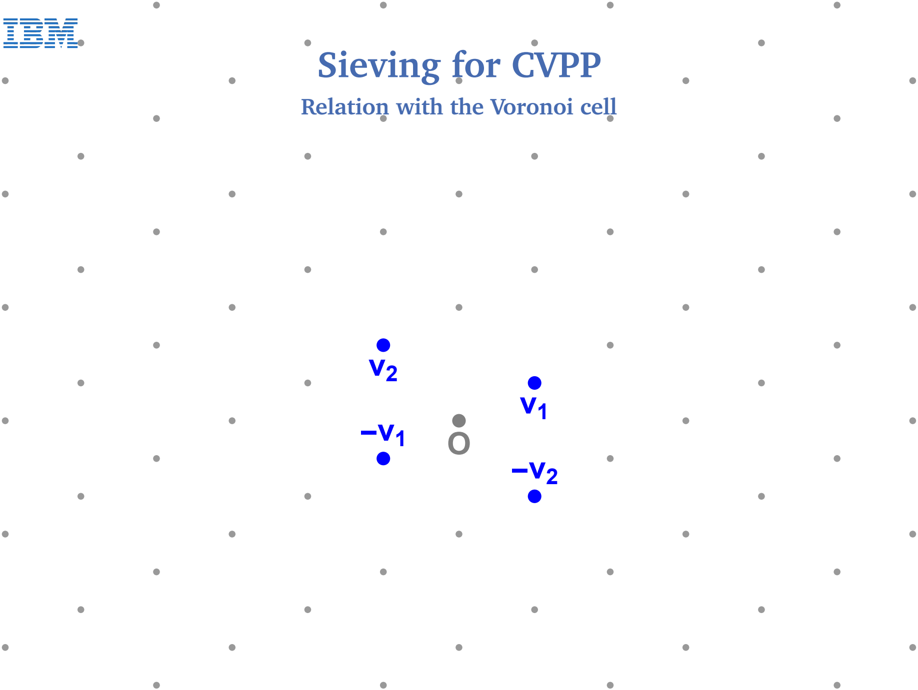

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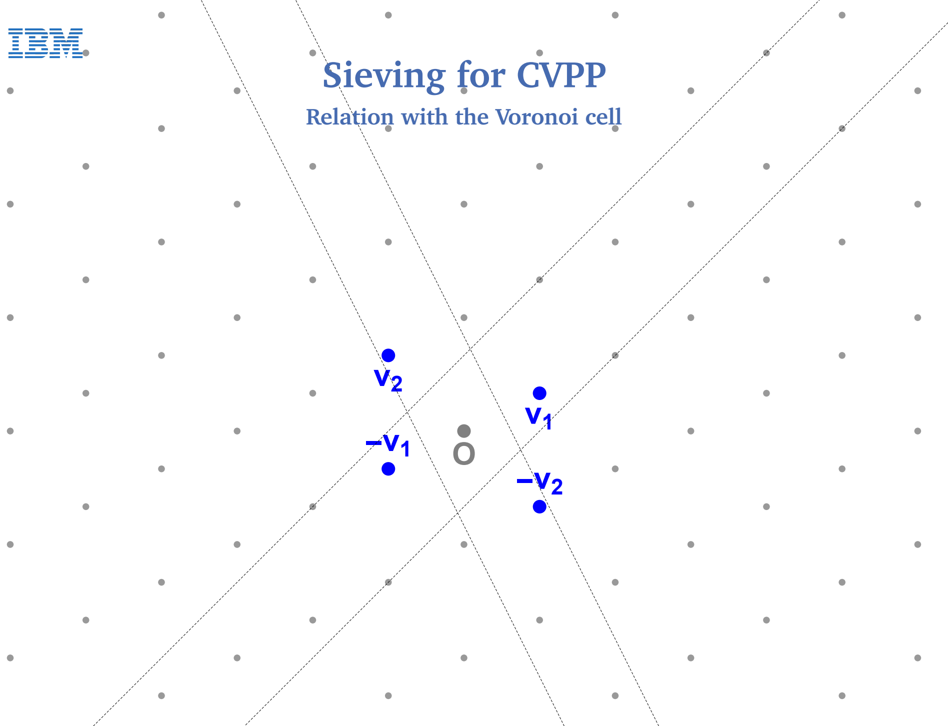

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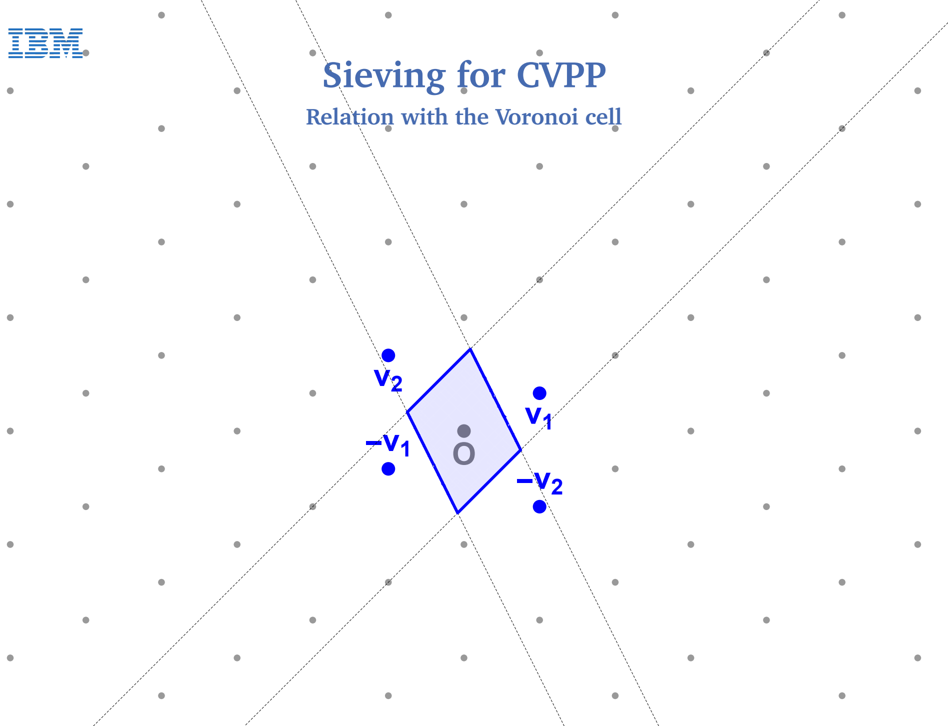

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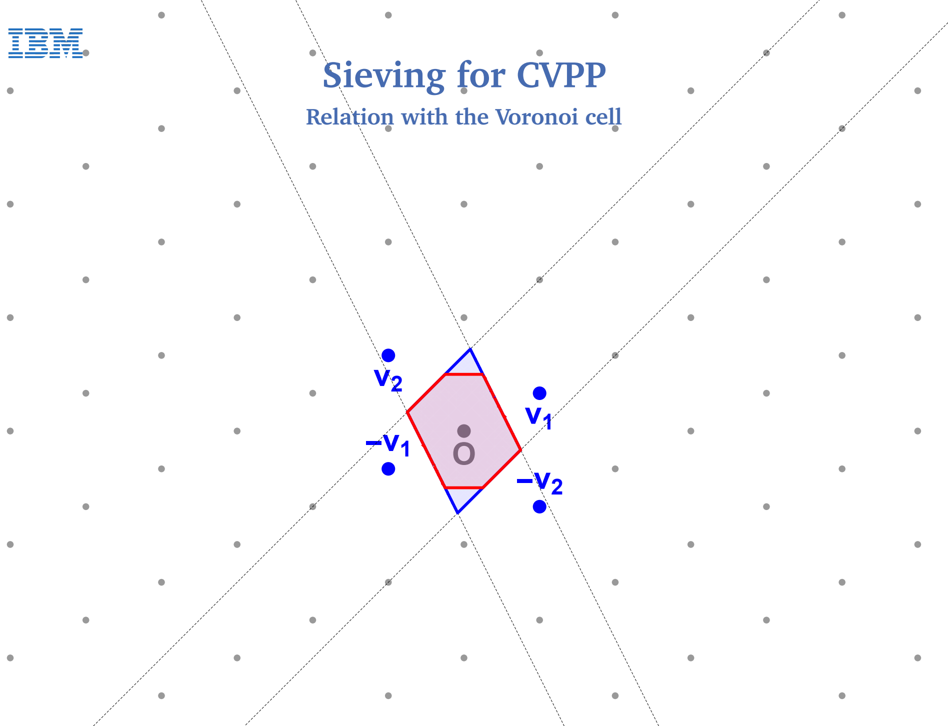

Sieving for CVPP

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  - Defined by $2^{0.21n+o(n)}$ short lattice vectors
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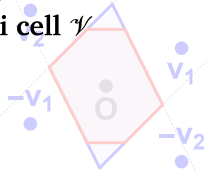

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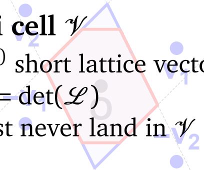

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  - Probability only over randomness of targets

Sieving for CVPP

Solving the problems

- Idea 1: **Larger lists, weaker reductions**

Sieving for CVPP

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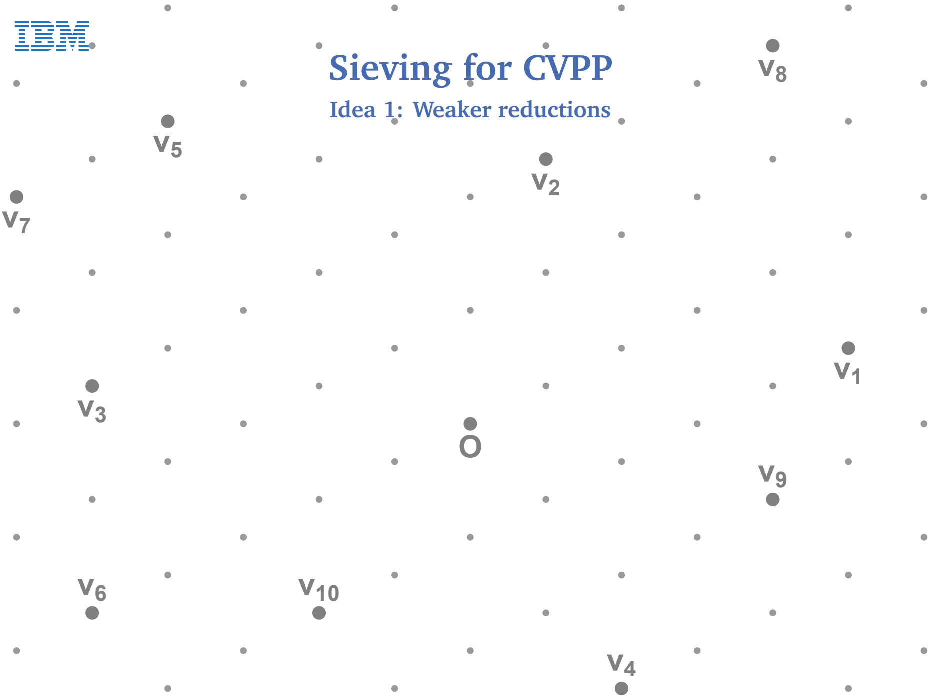
Sieving for CVPP

Solving the problems

- Idea 1: **Larger lists, weaker reductions**
  - Problem: Exponentially small success probability
  - To guarantee $\mathrm{Vol}(\mathscr{G}) \approx \mathrm{Vol}(\mathscr{V})$, need $2^{n/2+o(n)}$ vectors
  - Preprocessing: reduce $\boldsymbol{v}_1$ with $\boldsymbol{v}_2$ iff $\|\boldsymbol{v}_1 - \boldsymbol{v}_2\| \leq (\sqrt{2-\sqrt{2}})\|\boldsymbol{v}_1\|$
  - Fewer reductions $\Longrightarrow$ NNS techniques work even better!

Sieving for CVPP

Idea 1: Weaker reductions



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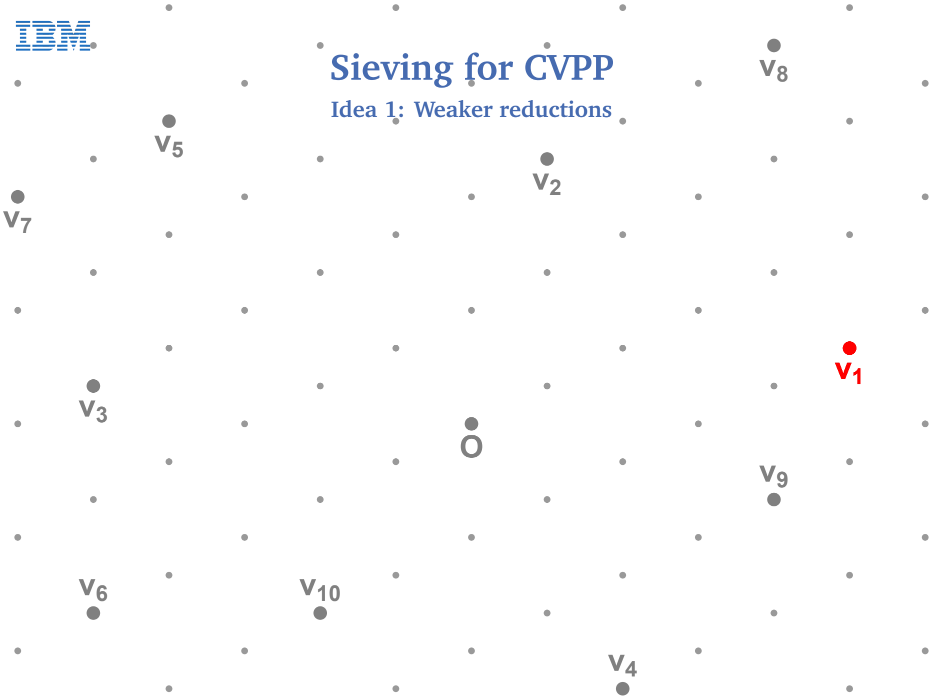

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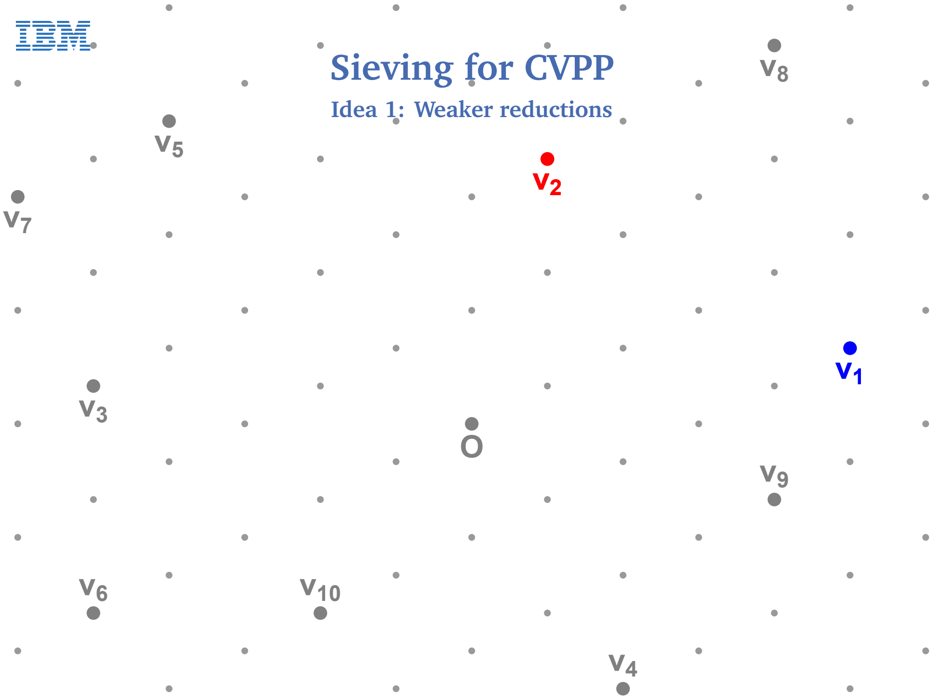

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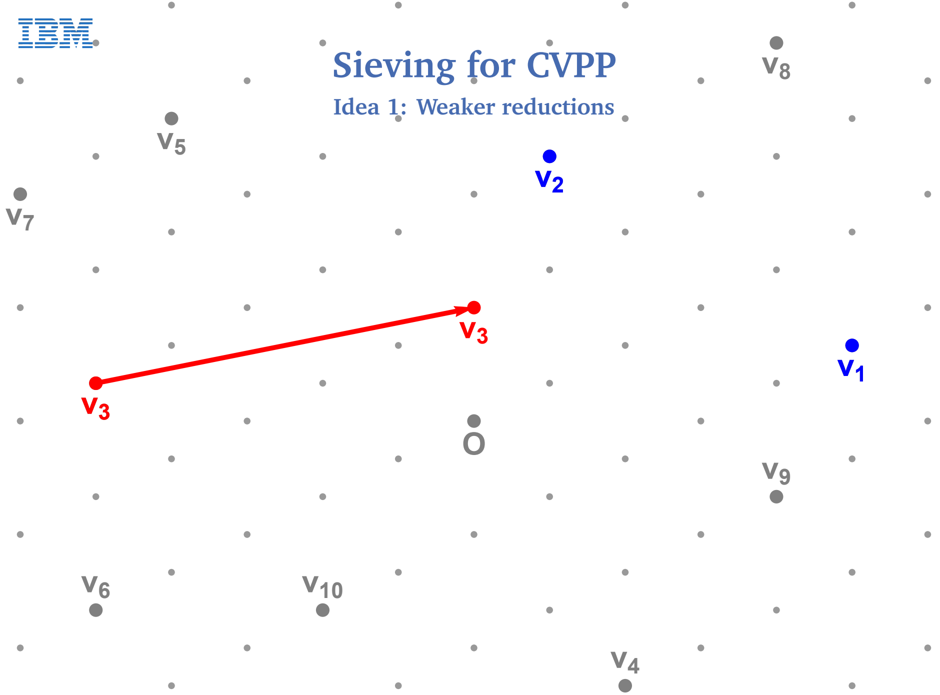

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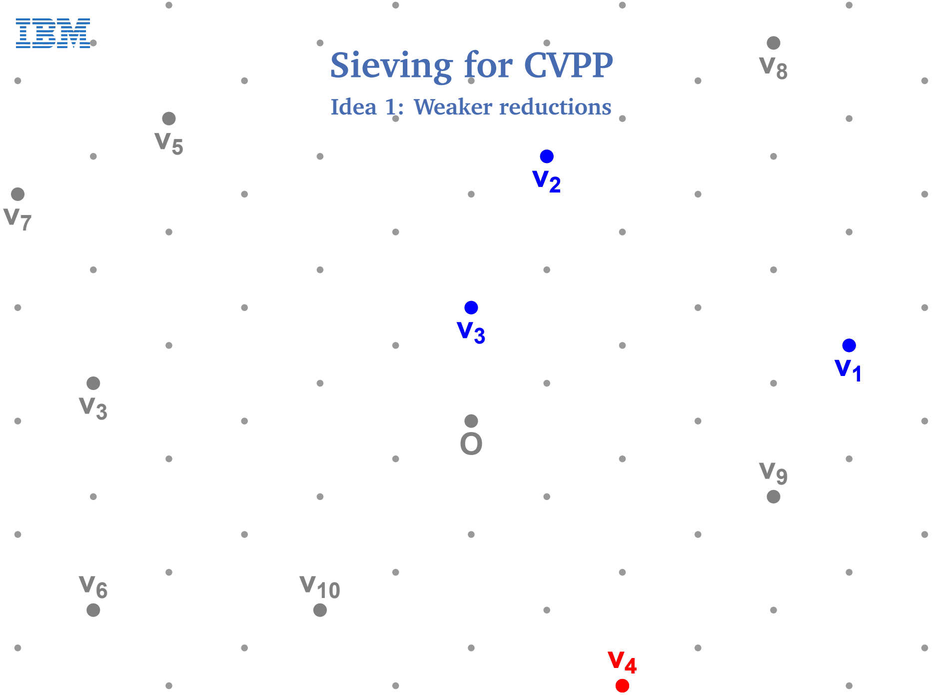

Sieving for CVPP

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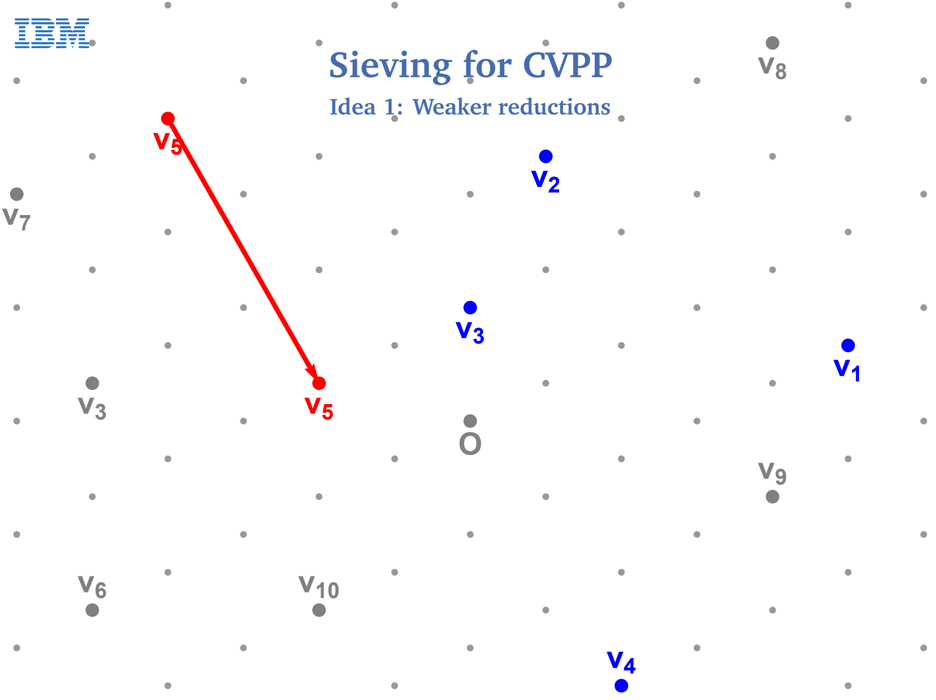

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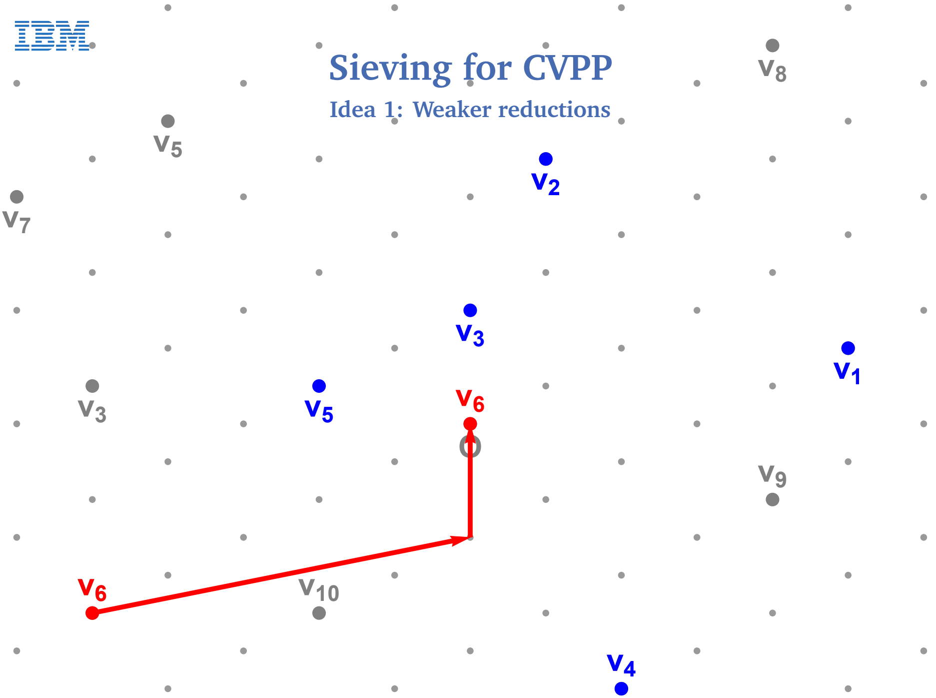

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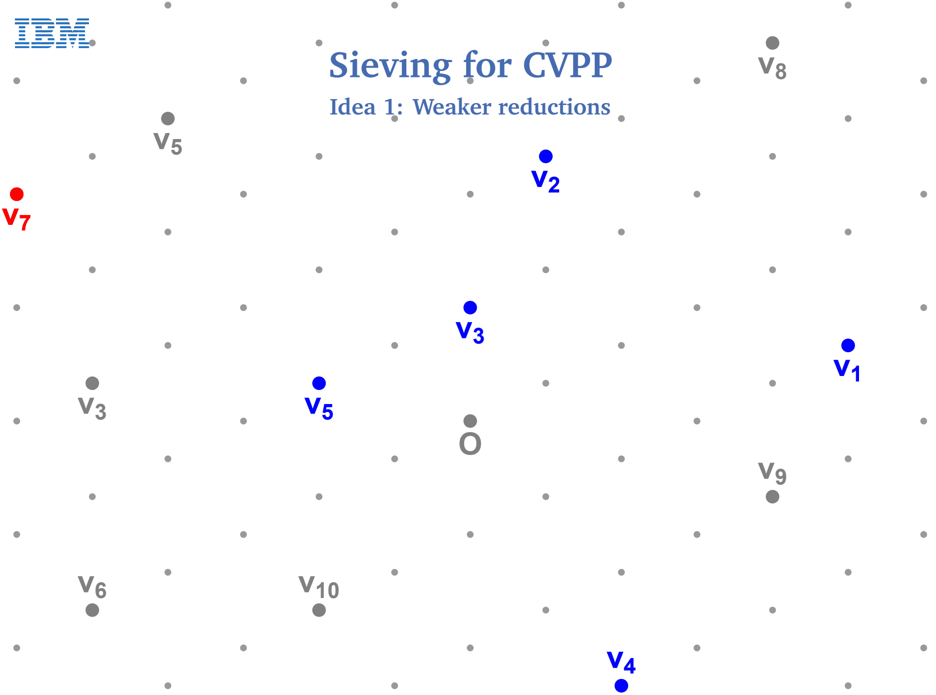

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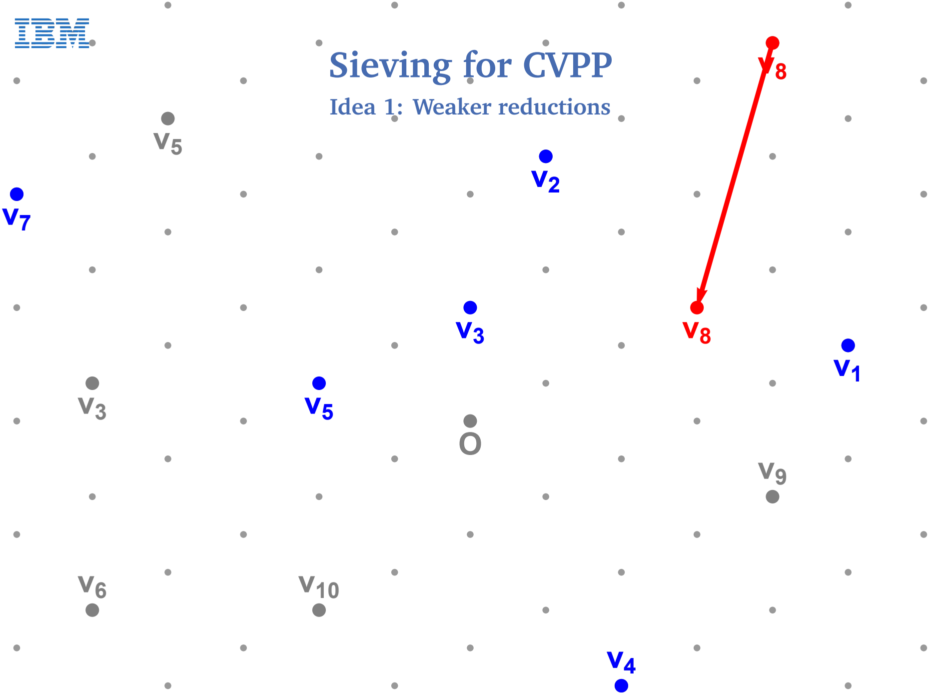

IBM

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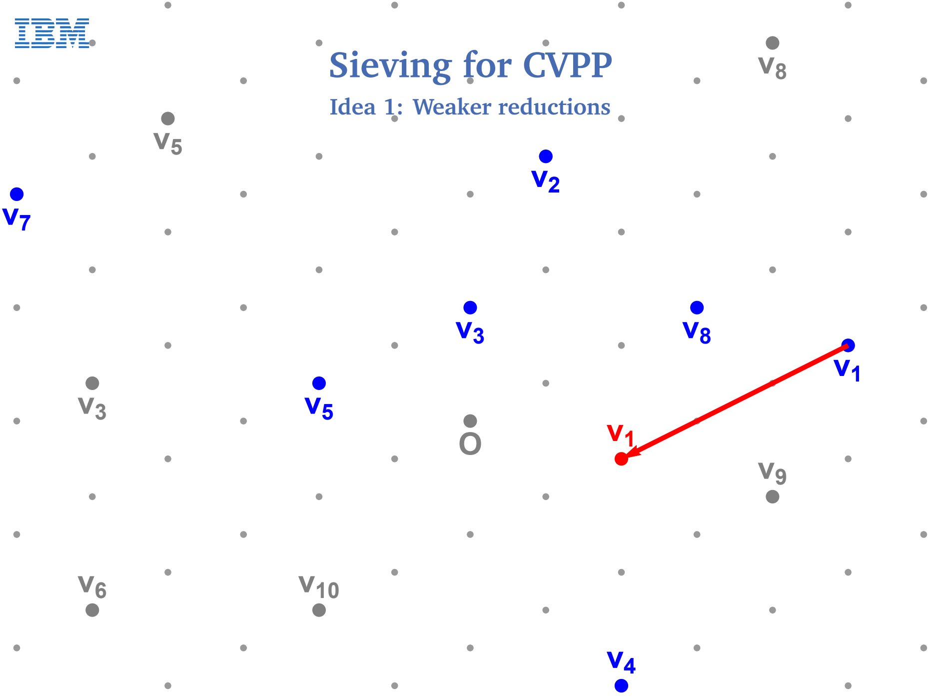

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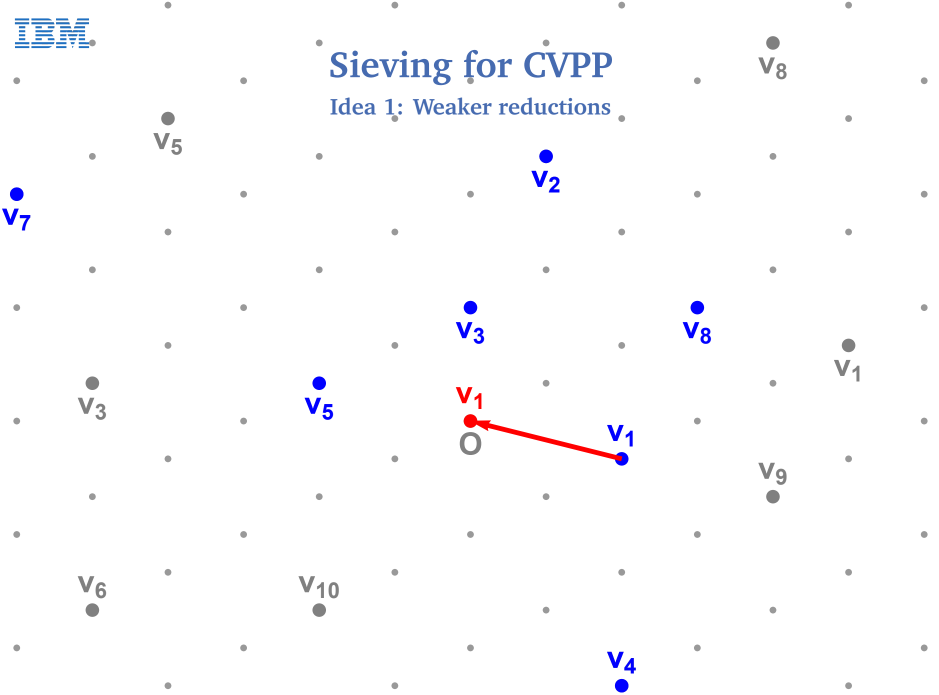

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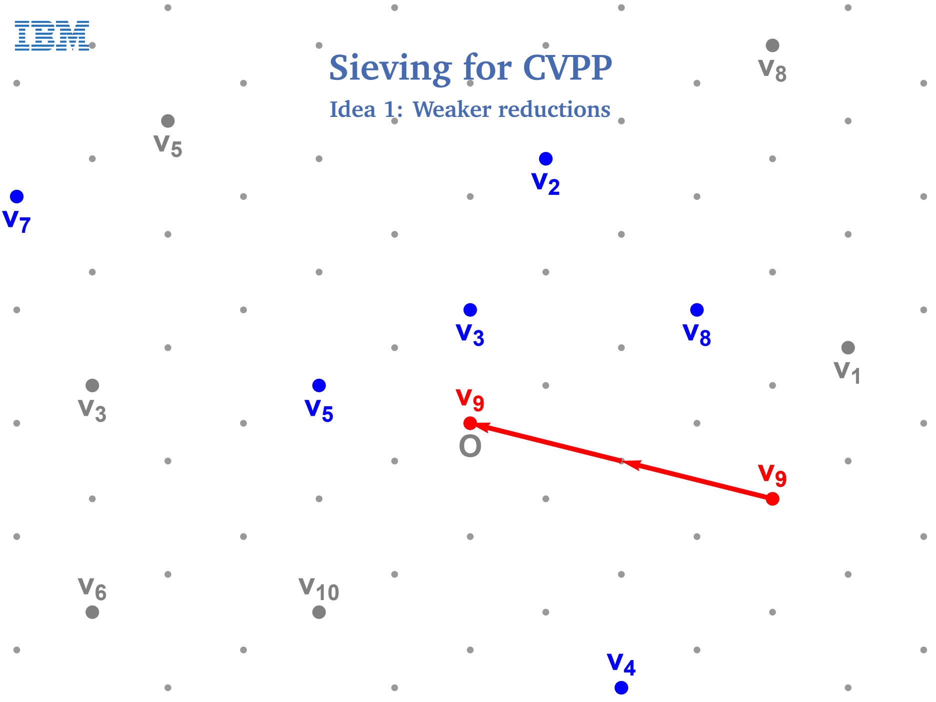

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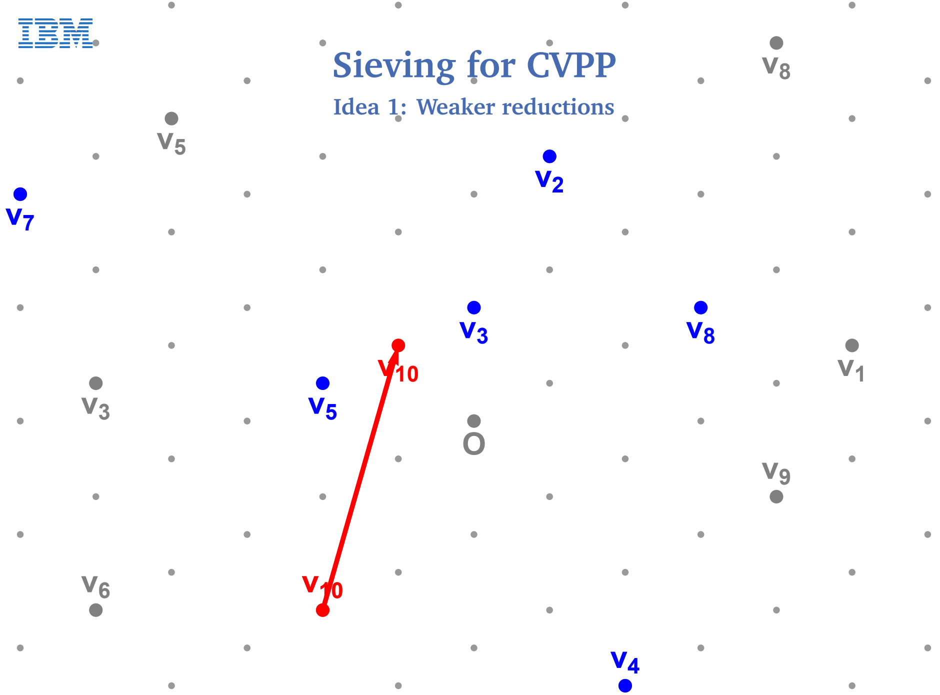

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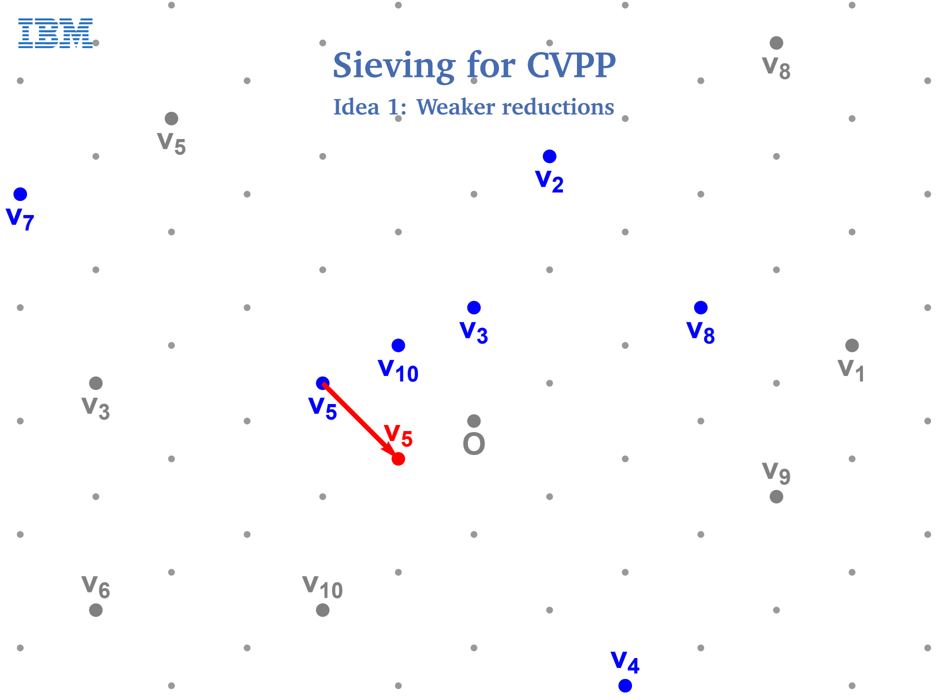

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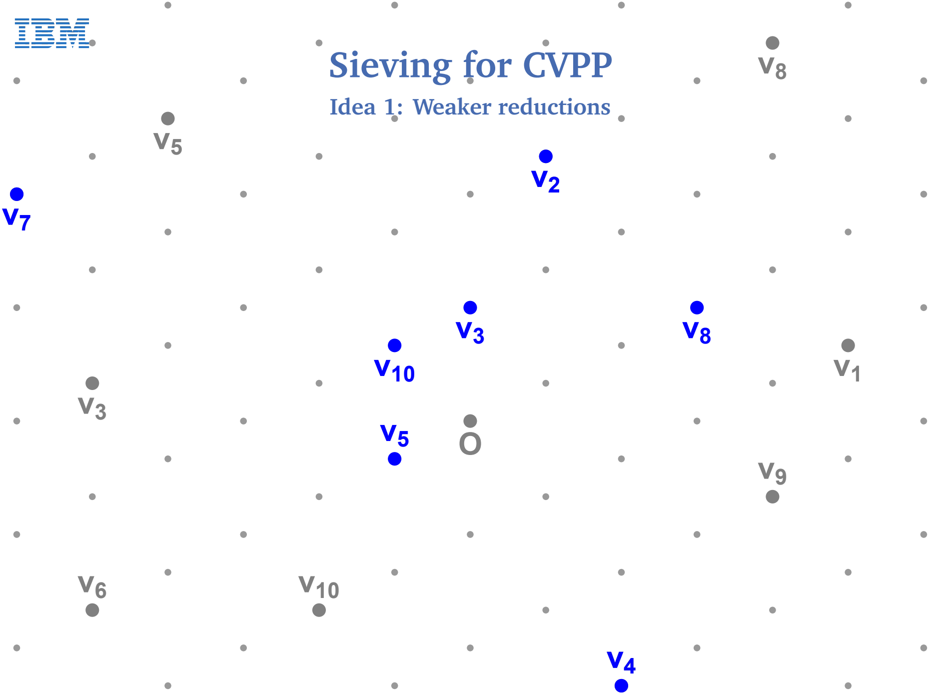

Sieving for CVPP

Solving the problems

- Idea 1: **Larger lists, weaker reductions**
  - Problem: Exponentially small success probability
  - To guarantee $\mathrm{Vol}(\mathscr{G}) \approx \mathrm{Vol}(\mathscr{V})$, need $2^{n/2+o(n)}$ vectors
  - Preprocessing: reduce $\boldsymbol{v}_1$ with $\boldsymbol{v}_2$ iff $\|\boldsymbol{v}_1 - \boldsymbol{v}_2\| \leq (\sqrt{2-\sqrt{2}})\|\boldsymbol{v}_1\|$
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  - Randomize target $\boldsymbol{t}$ before reducing ($\boldsymbol{t}' \in_R \boldsymbol{t} + \mathcal{L}$)
  - Randomness now over algorithm, independently of target
  - Optimize expected time (time / success probability)

Sieving for CVPP

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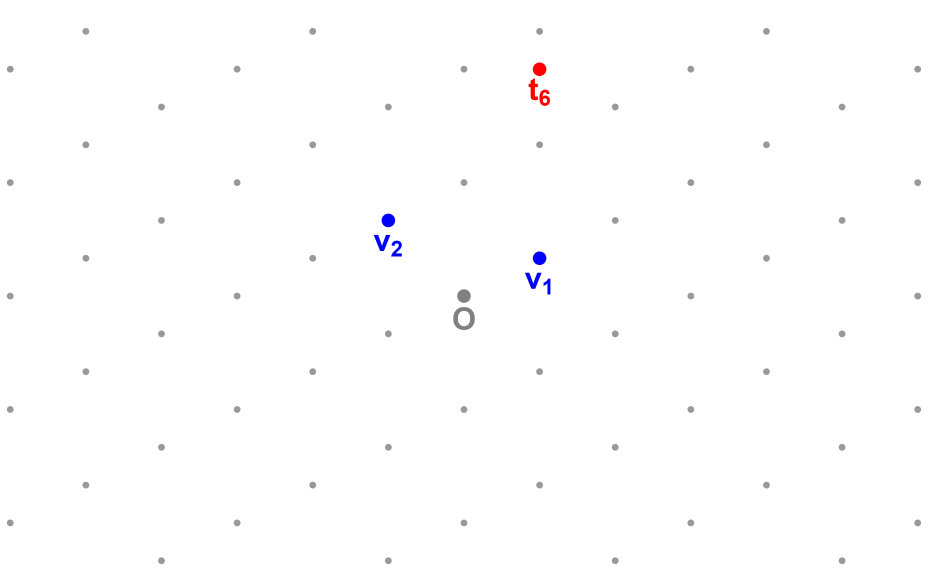

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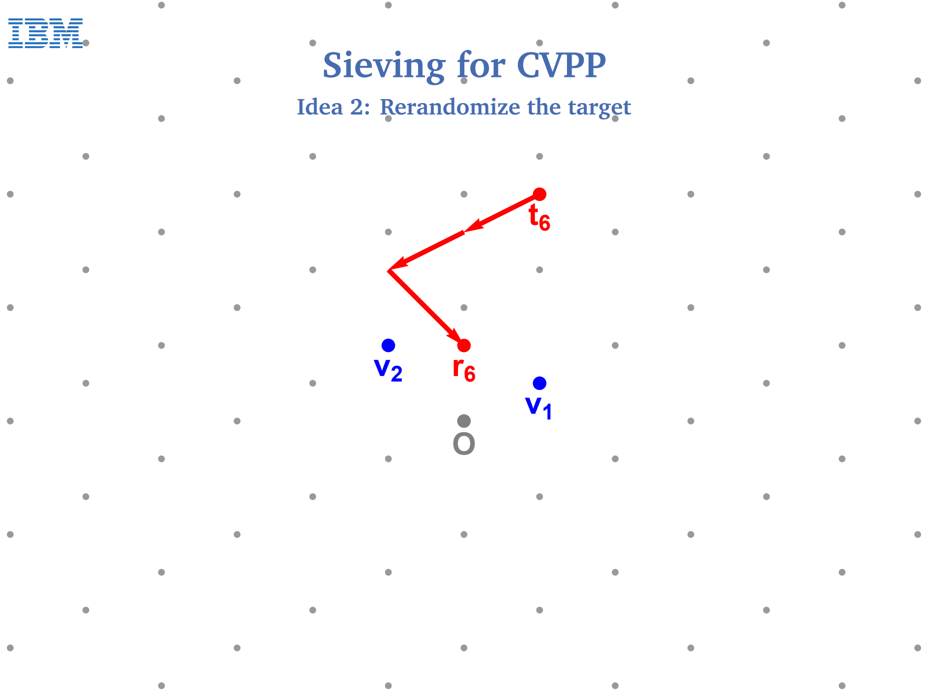

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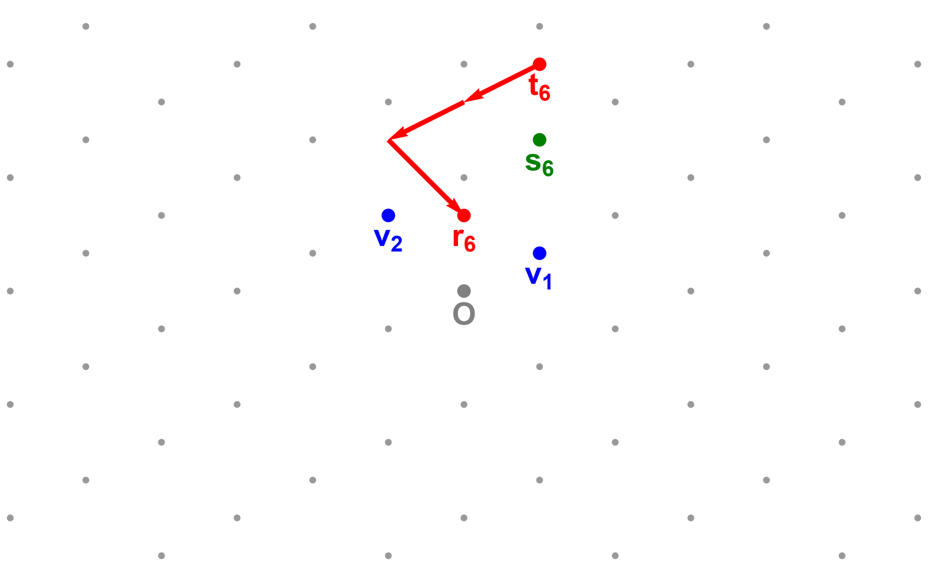

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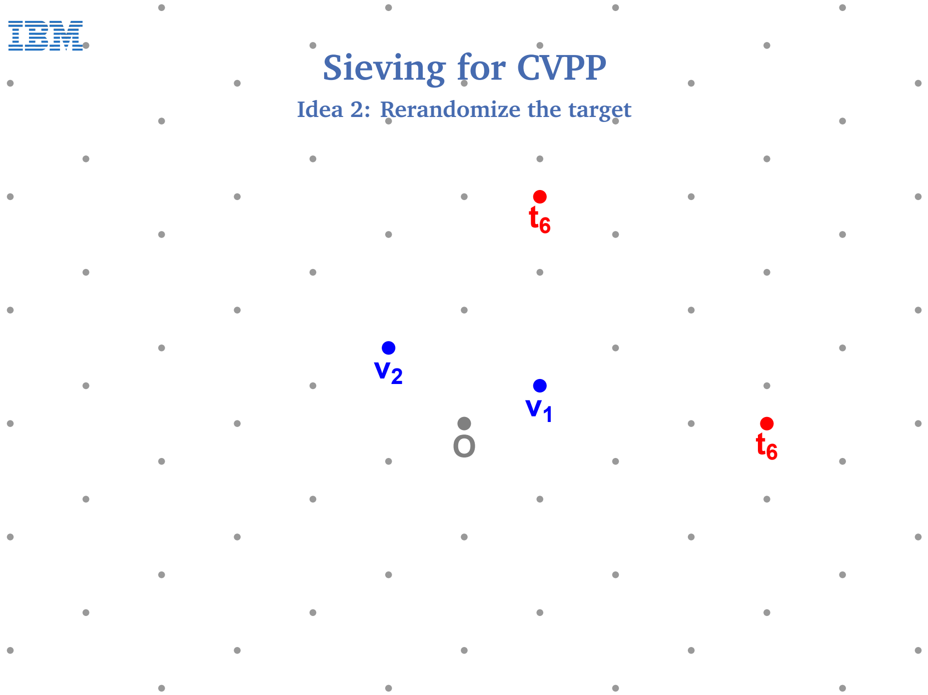

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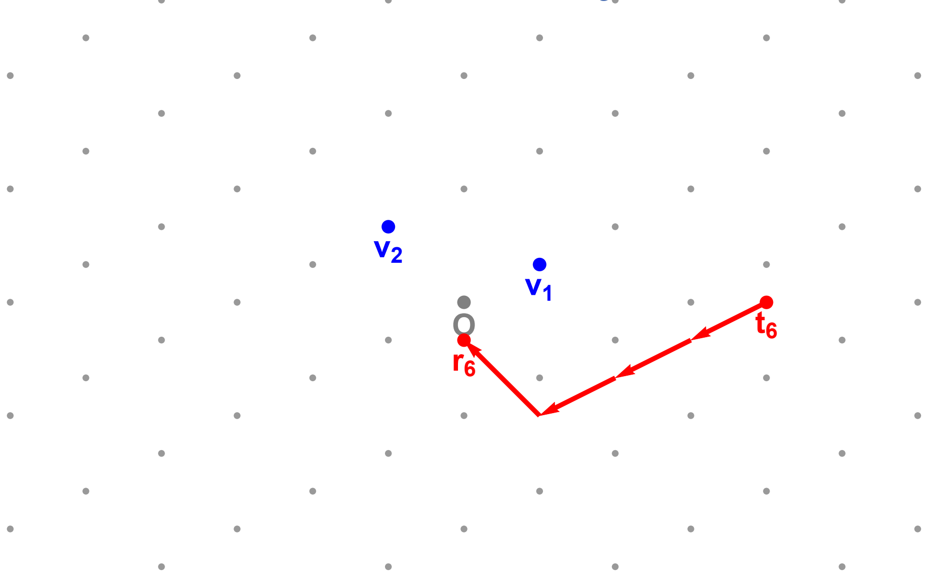

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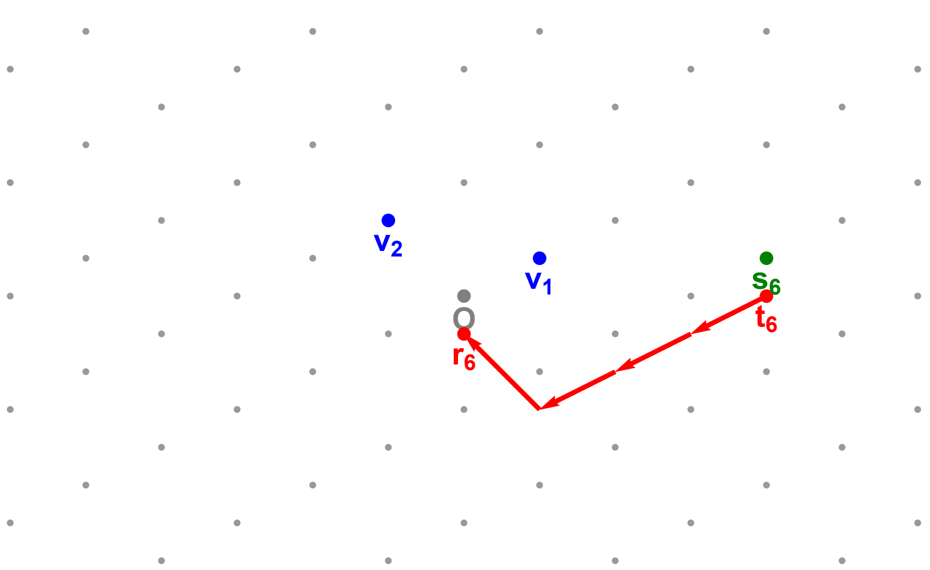
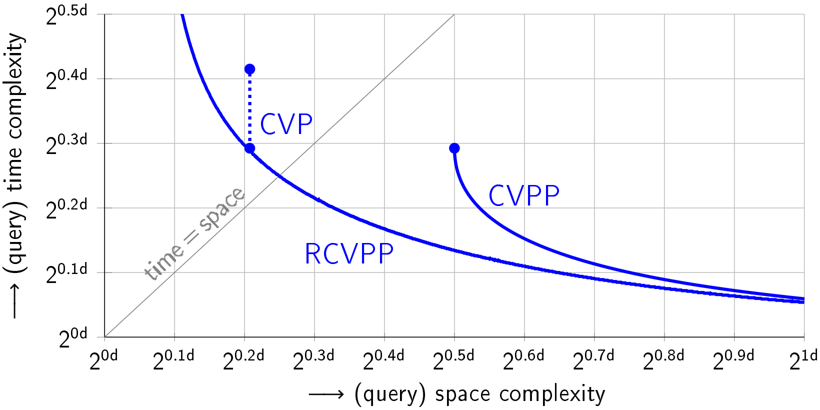


Sieving for CVPP

Trade-offs



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Questions?