KU LEUVEN

Generic Attacks on Lai-Massey Structures

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Selected Areas in Cryptography 2026

ESAT

COSIC

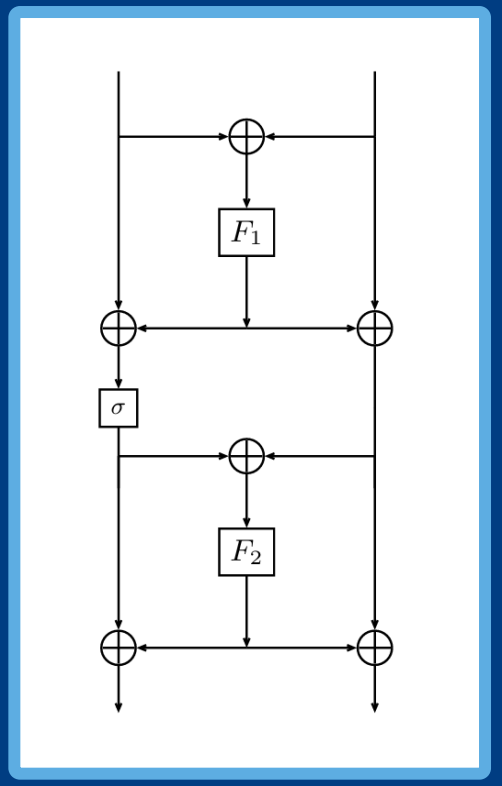
Outline

Part I: Evaluating Cipher Security

Multidimensional linear cryptanalysis

Part II: Our Contributions

Multidimensional linear distinguisher
Message recovery attack
Key recovery attack



Part I

Evaluating Cipher Security

Security testing and cryptanalysis

Linear Cryptanalysis

Multidimensional linear cryptanalysis

Linear Cryptanalysis: Linear Approximations

Finding probabilistic linear relations between plaintext, ciphertext, and key bits

$$\alpha.x \oplus \beta.F(x,k) = 0$$

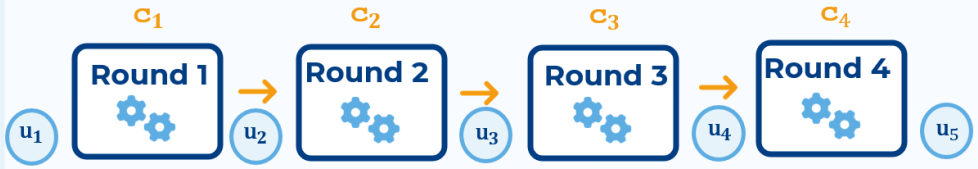
Holds with probability Pr

The Correlation

$$C(\alpha,\beta) = 2.(Pr[\alpha.x \oplus \beta.F(x) = 0] - \tfrac{1}{2})$$

± Correlation Sign

$$(-1)^{u^T k}$$



$$C_{total} = \prod c_i$$

Multiple Linear Cryptanalysis

Using **multiple linear approximations** simultaneously to improve attack efficiency

Why relying on a single linear approximation is insufficient for modern block ciphers?

Data Complexity

Multiple LC

Set of linear approximations:

$$(u_1, v_1), (u_2, v_2), \dots, (u_m, v_m)$$

How does it work?

- **Multiple linear approximations are evaluated together**
- **Each approximation contributes to the correlations**
- **Combined information improves distinguishing capability**

Lower data complexity!

"Linear Cryptanalysis Using Multiple Approximations"

Multidimensional Linear Cryptanalysis

Using multiple linear approximations that form a **vector space**

Why is relying on a single linear approximation insufficient for modern block ciphers?

Data Complexity

The Solution: Multidimensional LC

Vector Space Λ

A set of linear approximations:

$$(u, v) \in \Lambda$$

Multidimensional Analysis Process

- Define **the vector space** Λ of masks
- Calculate **correlations** for each mask and combine them
- Apply the test statistic to obtain a **distinguisher**
- **Possible extensions**: message and/or key recovery

"Multidimensional Linear Cryptanalysis of Reduced Round Serpent"

Multidimensional LC: Distinguisher

Known correlations vs **Unknown correlations**

Known Correlations

- LLR test
- Lower Data Complexity

Data Complexity:

$$q \approx \frac{1}{\mathrm{Cap}(\Lambda)}$$

Optimal
Known signs

Unknown Correlations

- χ^2 test
- Higher Data complexity

Data Complexity:

$$q \approx \frac{\sqrt{|\Lambda|}}{\mathrm{Cap}(\Lambda)}$$

Higher
Unknown signs

$$\mathrm{Cap}(\Lambda) = \Sigma c^2$$

Difference: $\sqrt{|\Lambda|}$ factor

"How Far Can We Go Beyond Linear Cryptanalysis?" and "An Experiment on DES Statistical Cryptanalysis"

Part II

Contributions

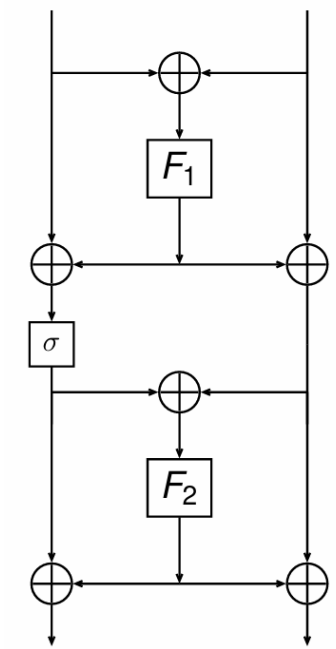
Generic multidimensional linear cryptanalysis of Lai-Massey structures

Roadmap

1. Multidimensional linear distinguisher
2. Message recovery attack
3. Key recovery attack

Lai-Massey Structures: The Concept

Lai-Massey Structures



Round functions

1. Chosen independently and randomly
2. Invertible
3. Key-dependent

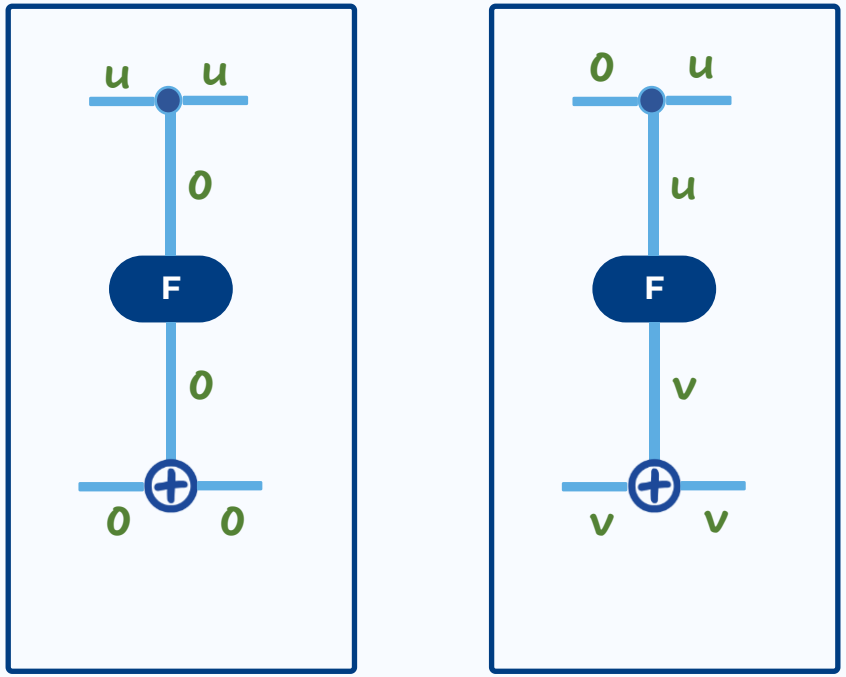
Time to analyze!

Lai-Massey Structures: The Core Idea

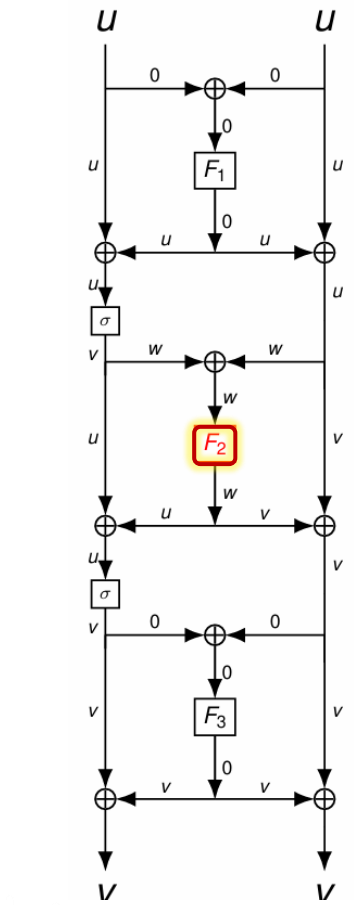
How does the attack work?

1. Invertible round functions
2. Non-zero mask → Non-zero mask
3. Key dependency

RESULT: Stronger statistical tests



Lai-Massey Structures : Distinguisher



What is the data-complexity?

1 **Vector space:** $w = u \oplus v$

$$\Lambda = \{((u,u),(v,v)) \mid w \in F_2^n\}$$

2 **Correlation of the trail:** $c \approx 2^{-n/2}$

$$C = \prod_{1}^{\lfloor r/2 \rfloor} c$$

3 **Capacity of the approximations:**

$$\text{Cap}(\Lambda) \approx 2^n C^2 \approx 2^{n-n\lfloor r/2 \rfloor}$$

χ^2 *test:* $q \approx 2^{n\lfloor r/2 \rfloor - n/2}$

Lai-Massey Structures: Distinguishing Attack

Data complexity for multidimensional distinguishing attacks

Data complexity

		Our results	Prior results*
3-round	KPA	$\approx 2^{n/2}$	$\approx 2^{n}$
4-round	CPA	$\approx 2^{n/2}$	$\approx 2^{n}$
5-round	KPA	$\approx 2^{3n/2}$	$\approx 2^{7n/4}$
6-round	CPA	$\approx 2^{2n}$	–

* “Pseudorandomness Analysis of the (extended) Lai-Massey Scheme” and “Generic Attacks on the Lai–Massey Scheme”

Part II

Contributions

Generic multidimensional linear cryptanalysis of Lai-Massey structures

Roadmap

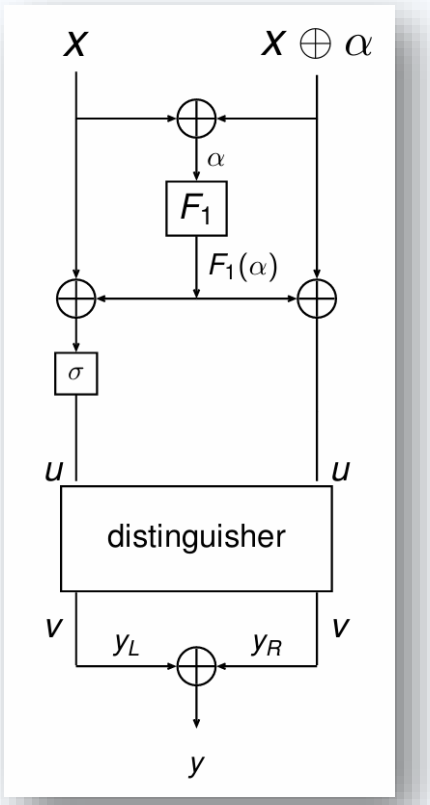
1 Multidimensional linear distinguisher

2 **Message recovery attack**

3 Key recovery attack

Lai-Massey Structures: Extension to Message Recovery

Extending multidimensional linear distinguishers to the message recovery attack



$\alpha_1 \stackrel{?}{=} \alpha_2$

3-round distinguisher

4-round message recovery*

$q \approx 2^{n/2}$

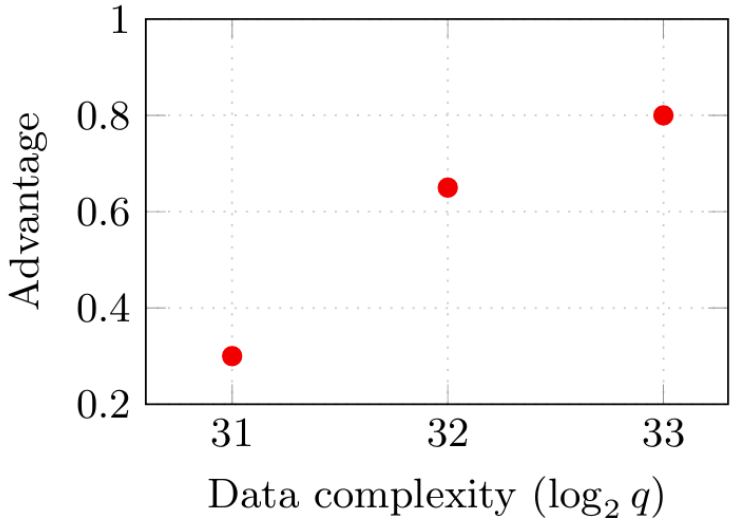
* “Message-recovery attacks on Feistel-based format preserving encryption”

Lai-Massey Structures: Experiment on SPC

Two-sample χ^2 test

$$f_1[E_k(x^i, x^i \oplus \alpha_1)]$$
$$f_2[E_k(x^j, x^j \oplus \alpha_2)]$$

for $i = 0$ *to* $q/2$ **do**
$$s \leftarrow \sum_i \frac{(f_1(z_i) - f_2(z_i))^2}{f_1(z_i) + f_2(z_i)};$$
end



Part II

Contributions

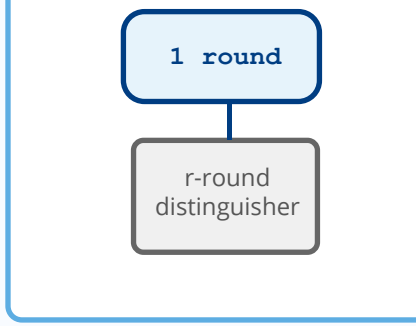
Generic multidimensional linear cryptanalysis of Lai-Massey structures

Roadmap

1. Multidimensional linear distinguisher
2. Message recovery attack
3. **Key recovery attack**

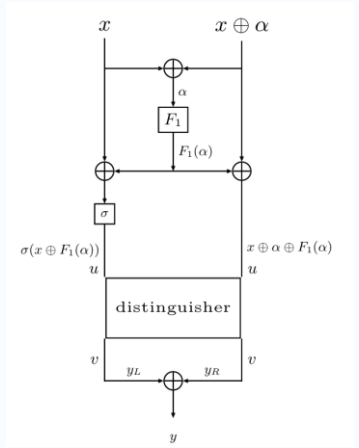
Feistel Structures: Key Recovery

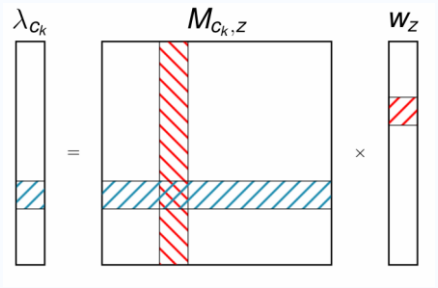
Extending multidimensional linear distinguishers to recover the secret key



Compute statistic

Matrix-vector product using FFT





Round key (n) **Block Size (n)** **Rounds (r)**

FFT Optimization*

$\mathcal{O}(n2^n + q)$

Matrix-vector product

Data complexity

Naive method

$\mathcal{O}(q2^n)$

Matsui's method

$\mathcal{O}(n2^{2n} + q)$

FFT optimization significantly reduces time complexity

* "Improving the Time Complexity of Matsui's Linear Cryptanalysis"

Lai-Massey Structures: Complexity bounds

Distinguishing and key-recovery attack complexity bounds for GFC

$$\mathcal{O}\left(n\,2^{r'n} + q\right)$$

$q \approx 2^{n/2}$

Distinguishing attack rounds

$r = 3$

Key recovery attack rounds

$r + r' = 4, \quad K = 2n \;\; and \;\; T = n2^{n}$

$r + r' = 6, \quad K = 4n \;\; and \;\; T = n2^{3n}$

$q \approx 2^{3n/2}$

Distinguishing attack rounds

$r = 5$

Key recovery attack rounds

$r + r' = 6, \quad K = 2n \;\; and \;\; T = n2^{n}$

$r + r' = 8, \quad K = 4n \;\; and \;\; T = n2^{3n}$

5-round distinguisher can be extended to an 8-round key recovery attack

Conclusion

Generic Multidimensional linear cryptanalysis of Lai-Massey structures:

1 **Multidimensional linear distinguisher**

6-round generic distinguishing attack in the CPA model

2 **Message recovery attack**

Message recovery attack with the same data complexity as the distinguisher

3 **Key recovery attack**

8-round key-recovery attack

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Multidimensional Linear Distinguishers

Applying Theorem 1 to the vector space $V = \Lambda$ with the random variable $\mathbf{z} = (\mathbf{x}, F(\mathbf{x}))$, where $\mathbf{x}$ is uniformly random over $\mathbb{F}_2^n$ and is a function $F : \mathbb{F}_2^n \to \mathbb{F}_2^n$, yields the following result:

$$\Pr\left((\mathbf{x}, F(\mathbf{x})) = (t_1, t_2) \bmod \Lambda^{\perp}\right) = \frac{1}{|\Lambda|} \sum_{(u,v)\in\Lambda} (-1)^{u^T t_1 + v^T t_2} C^F_{v,u}. \tag{1}$$

where $\Lambda^{\perp} = \left\{(\mathbf{x}, F(\mathbf{x})) \in \mathbb{F}_2^n \times \mathbb{F}_2^n \mid u^T\mathbf{x} + v^T F(\mathbf{x}) = 0 \text{ for all } (u, v) \in \Lambda\right\}$. In other words, the correlations $C^{\mathrm{F}}_{v,u}$ for $(u, v) \in \Lambda$ completely determine the probability distribution of the linear projections of plaintext and ciphertext, which defines a linear map from $\mathbb{F}_2^n \times \mathbb{F}_2^n$ to the quotient space $\mathbb{F}_2^n \times \mathbb{F}_2^n/\Lambda^{\perp}$.

Multidimensional Linear Distinguishers

Let V be a subspace of $\mathbb{F}_2^n$.

1. **Quotient Space.** The *quotient space* $\mathbb{F}_2^n/V$ is defined as

$$\mathbb{F}_2^n/V = \{x + V \mid x \in \mathbb{F}_2^n\},$$

where $x + V = \{x + v \mid v \in V\}$. There is a linear map (projection)

$$\pi_V : \mathbb{F}_2^n \to \mathbb{F}_2^n/V, \quad x \mapsto x + V,$$

and for any $x, y \in \mathbb{F}_2^n$, we have

$$x \equiv y \pmod{V} \iff \pi_V(x) = \pi_V(y) \iff x - y \in V.$$

2. **Orthogonal Complement.** The *orthogonal complement* of V is defined as

$$V^\perp = \{x \in \mathbb{F}_2^n \mid x^\top v = 0, \ \forall v \in V\}.$$

Multidimensional χ^2 Test and Data Complexity

Expected value for the real cipher:

$$\mu_{\text{real}} = \mathbb{E}[\chi^2_{\text{real}}] \approx |\Lambda| - 1 + q \sum_{(u,v)\in\Lambda\setminus\{(0,0)\}} \mathbb{E}\left[(C^F_{v,u})^2\right]$$

Expected value and variance for the ideal cipher:

$$\mu_{\text{ideal}} = |\Lambda| - 1, \qquad \sigma^2_{\text{ideal}} = 2(|\Lambda| - 1)$$

- Successful distinguisher condition:

$$\mu_{\text{real}} - \mu_{\text{ideal}} \gg \sigma_{\text{ideal}}$$

- Substituting yields:

$$q \sum_{(u,v)\in\Lambda\setminus\{(0,0)\}} \mathbb{E}\left[(C^F_{v,u})^2\right] \gg \sqrt{|\Lambda|}$$

- Using the **capacity definition**:

$$q \approx \frac{\sqrt{|\Lambda|}}{\text{Cap}(\Lambda)}$$

By the central limit theorem, the $\boldsymbol{\chi}^2_{\text{ideal}}$ asymptotically approaches the normal distribution $\mathcal{N}(\nu, 2\nu)$ as 2^n and q approach infinity. The false-positive rate is calculated by

$$P_F = \Pr\left[\chi^2_{\text{ideal}} \geq t\right] = \Pr\left[\underbrace{\frac{\chi^2_{\text{ideal}} - \nu}{2\nu}}_{\mathbf{Z} \sim \mathcal{N}(0,1)} \geq \frac{t-\nu}{2\nu}\right] = 1 - \Phi\left[\frac{t-\nu}{2\nu}\right],$$

where $\Phi(x)$ is the cumulative distribution function. Hence, the threshold value t is $t = \nu + \sqrt{2\nu}\,\Phi^{-1}(1 - P_{\text{F}})$. Once t is set for a fixed P_{F}, the success probability can be calculated by how often the χ^2_{real} statistic (the χ^2 statistic based on samples from the real cipher) exceeds t for a given data complexity.

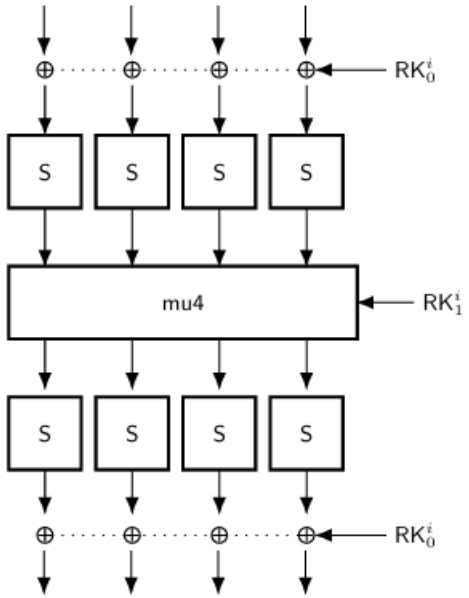


Fig. 2: The round function of FOX64.

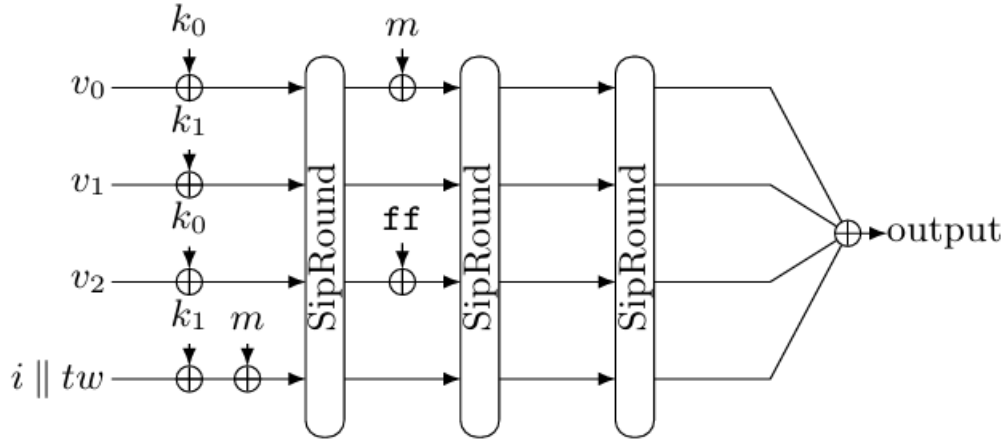
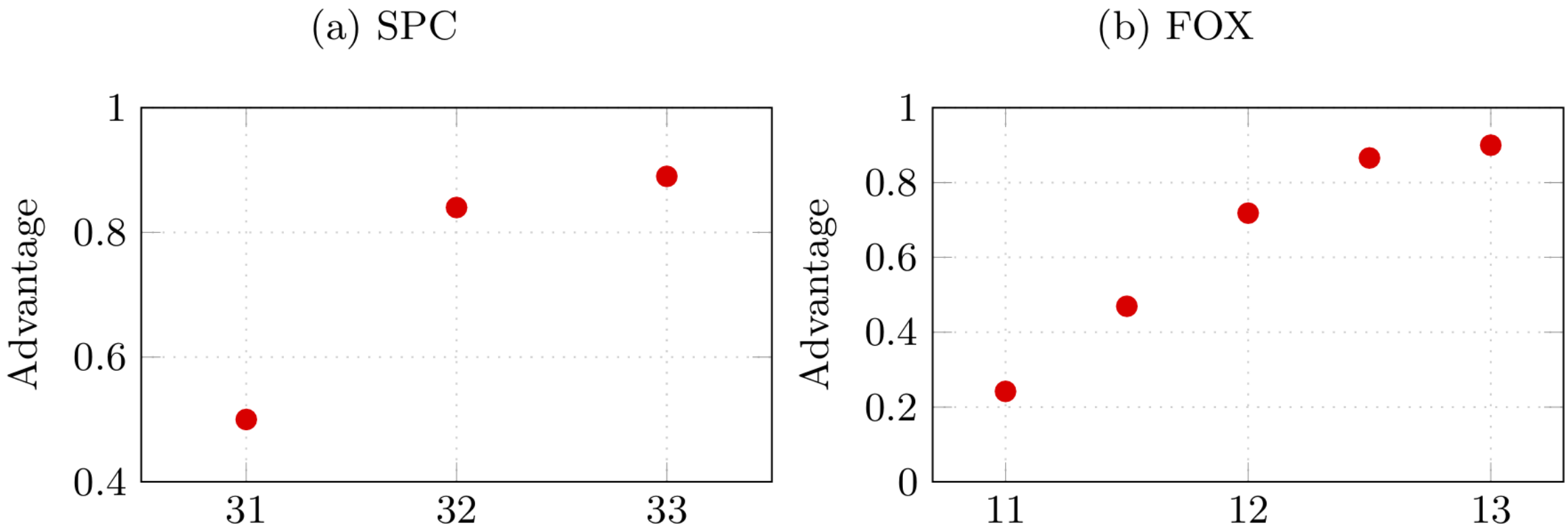
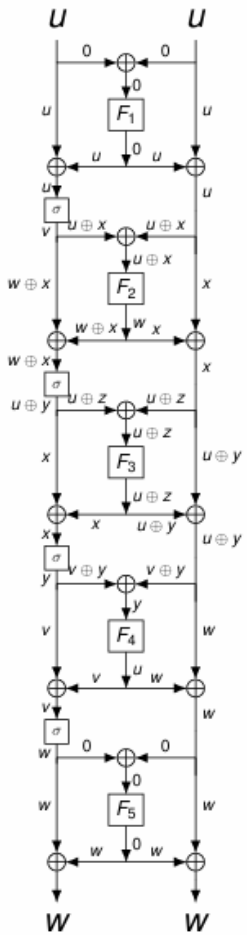


Fig. 3: The round function of SPC.

Fig. 8: Advantage as a function of data-complexity for 3-round χ^2 distinguishers where $P_{\mathrm{F}} = 0.1$.



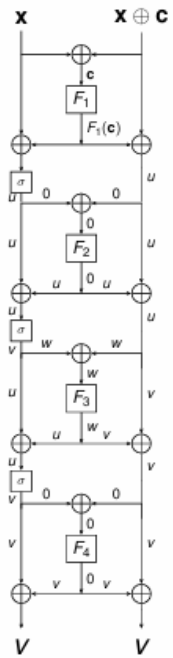
5-round linear trail with 3 active round functions

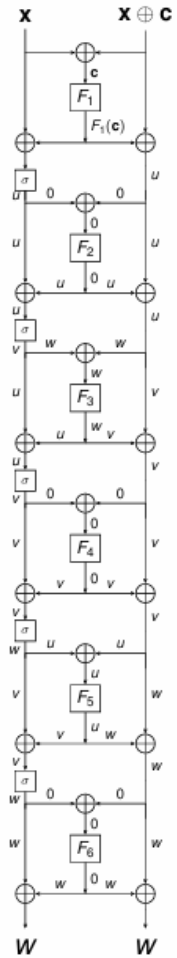


5-Round Lai–Massey Linear Trails

- Case r = 5 needs special attention:
- Direct formula gives $E(c^2(w)) = 2^{-2n}$, usually too small for a distinguisher.
- Additional linear trails exist with identical input/output masks.
- Example trail activates only F_2, F_3, F_4 while F_1 and F_5 stay inactive.
- Permutation consistency: $y = \sigma^{-1}(x)$, $z = \sigma^{-1}(y)$, $x = \sigma^{-1}(z)$ with $x \oplus y = z$.
- 2^n possible x values → extra trails with squared correlation $\approx 2^{-3n}$.
- Total expected squared correlation:
- $E(c^2(w)) \approx 2^{-2n} + 2^n \cdot 2^{-3n} \approx 2 \cdot 2^{-2n}$.
- For $r \leq 4$ multiplicity negligible; for r = 5 correlation doubles.

Generic multidimensional linear distinguisher under CPA model





$$q = \mathcal{O}\left(2^{n\lfloor (r-1)/2 \rfloor - n/2}\right)$$

Generic key recovery attack

$\mathcal{O}(n2^{n(r'-1)\ell}+q+2^{\ell})$

$$T = \begin{cases} \mathcal{O}\left(n2^{n+(r'-1)\ell} + 2^{\ell}\right), & r = 3, \\ \mathcal{O}\left(n2^{n+(r'-1)\ell} + 2^{3n/2} + 2^{\ell}\right), & r = 5, \end{cases}$$

$$\mathcal{O}(n2^{nr'}), \quad \ell = n,$$
$$\mathcal{O}(n2^{2nr'-n} + 2^{2n}), \quad \ell = 2n$$

$r = 3$

$$\mathcal{O}(n2^{nr'} + 2^{3n/2}), \quad \ell = n,$$
$$\mathcal{O}(n2^{2nr'-n} + 2^{2n}), \quad \ell = 2n$$

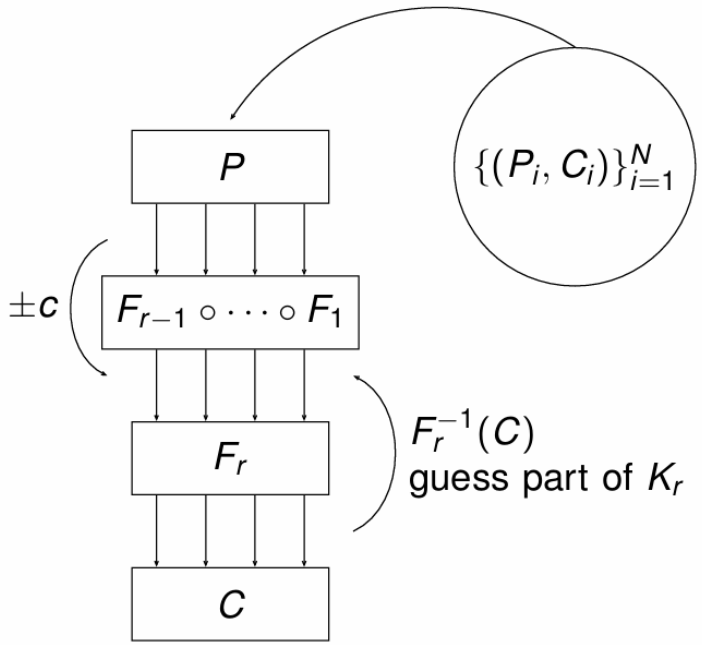
$r = 5$

r	ℓ	K	$r'_{\max}$	T
3 or 5	n	$2n$	1	$n2^n$
		$4n$	3	$n2^{3n}$
	$2n$	$4n$	2	$n2^{3n}$

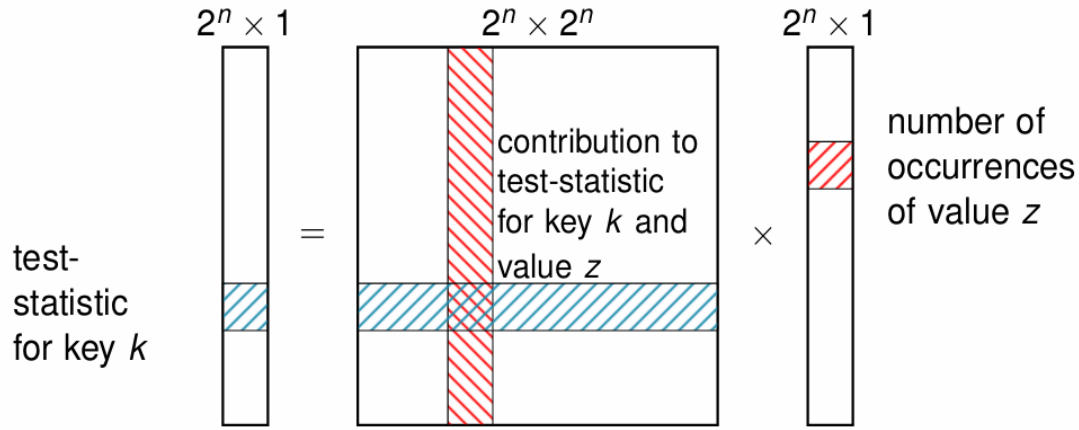
Table 5: Key recovery results of $r + r'$-round FOX64.

$r + r'$	r'	Data	Ref.	Work effort
4	1	$2^{15.8}$	§ 5.1	2^{64}
		2^{9}	[33]	$2^{45.4}$
5	2	$2^{16\,\dagger}$	§ 5.1	2^{101}
		2^{9}	[33]	$2^{109.4}$
6	1	$2^{32\,\dagger}$	§ 5.1	2^{64}
		15	[14]	2^{124}
7	2	$2^{32\,\dagger}$	§ 5.1	2^{101}
		$2^{30.9}$	[14]	2^{124}

† Asymptotic results.



$\mathcal{O}(q2^n)$



Matsui's Method

$\mathcal{O}(q + 2^{2n})$

FFT-based Method

$\mathcal{O}(q + n2^n)$